

Remarks on Exact RG Equations

H. Osborn¹ and D.E. Twigg²

Department of Applied Mathematics and Theoretical Physics,
Wilberforce Road, Cambridge, CB3 0WA, England

Abstract

Exact RG equations are discussed with emphasis on the role of the anomalous dimension η . For the Polchinski equation this may be introduced as a free parameter reflecting the freedom of such equations up to contributions which vanish in the functional integral. The exact value of η is only determined by the requirement that there should exist a well defined non trivial limit at an IR fixed point. The determination of η is related to the existence of an exact marginal operator, for which an explicit form is given. The results are extended to the exact Wetterich RG equation for the one particle irreducible action Γ by a Legendre transformation. An alternative derivation of the derivative expansion is described. An application to $\mathcal{N} = 2$ supersymmetric theories in three dimensions is described where if an IR fixed point exists then η is not small.

¹ho@damtp.cam.ac.uk

²det28@cam.ac.uk

1 Introduction

Historically the appreciation of the role of the renormalisation group, which corresponds to varying the cut off scale, was essential to the modern understanding of quantum field theories. As a consequence it becomes clear that quantum field theories, which need a cut off in their formulation, must be considered as belonging to an infinite dimensional space which may be parameterised in terms of couplings associated with all scalar operators consistent with the basic symmetries of the theory. The RG equations then describe flows in this space of quantum field theories under changes in the cut off scale Λ , the flow being determined by the requirement that physical observables are independent of the cut off, at least in the neighbourhood of fixed points, for energies much less than the cut off. This allows a continuum quantum field theory to be obtained which satisfies the required symmetries and is independent of the choice of cut off. A particular realisation may be obtained in terms of renormalisable theories where the space of quantum field theories is restricted to a submanifold, usually finite dimensional, which is invariant under RG flow. For renormalisable theories the RG flow equations are linear and the flow in the space of couplings is given in terms of the associated β -functions.

A particular realisation of the renormalisation group, for quantum field theories where a cut off may be introduced through a modification of the quadratic kinetic term, is obtained through exact functional non linear RG equations, due to Wilson [1] and developed in various alternative forms in [2, 3, 4]. Reviews covering different aspects are found in [5, 6, 7, 8, 9, 10, 12]. Such exact RG equations provide an in principle non perturbative definition of RG flow. Nevertheless such equations are restricted, at least in most applications, to theories containing just scalar fields, although extensions to fermion fields are relatively straightforward. The exact RG equations are also hard to approximate in a consistent fashion which allows calculable higher order corrections. Although it is possible to recover perturbative results, and rederive the perturbative results for β -functions, the methods involved tend to be distinct from those used in non perturbative approximations. A particular issue is connected with the anomalous dimension η of the basic scalar fields in the theory. Although η may be introduced as an additional parameter in the RG flow equations, where it is essentially arbitrary, the precise way in which it is to be determined is not always fully resolved.

The aim in this paper is to analyse such issues in more detail than hitherto. The RG equations determined how the theory varies with t , where $t \sim -\ln \Lambda$. We argue that η is an arbitrary parameter until the we consider limit $t \rightarrow \infty$. The requirement that there exists a well defined limit, with long range order, when $t \rightarrow \infty$ is the necessary and sufficient condition to determine η . In particular as was discussed by Wegner [3], and emphasised more recently by Rosten [10], the existence of discrete values for η is linked to the presence of an exact marginal operators \mathcal{Z} . In this case \mathcal{Z} generates a line of equivalent fixed points which corresponding essentially to an overall rescaling of the fields. The limit $t \rightarrow \infty$ may also generate a trivial, so called high temperature, fixed point with no long range order but then there is no marginal operator and η is not determined.

In the next section we discuss in detail various aspects of the exact RG equations as introduced by Wilson and its later alternative, with an essentially similar form, developed

by Polchinski [4]. Although the analysis is not rigorous we attempt to make precise the mathematical framework and avoid any particular choice of the cut off function necessary in the derivation of the RG flow equation. As usual for simplicity we consider a single scalar field φ which is rescaled by the cut off to be dimensionless and consider the RG flow of the action functionals $\mathcal{S}_t[\varphi]$ as well as their IR fixed points $\mathcal{S}_*[\varphi]$. The original Wilson/Polchinski equations are non linear but there is a transformation such that $\mathcal{S}_t[\varphi] \rightarrow T[\varphi_t]$ that linearises the RG flow equation. The transformation is generated by the action of a functional operator $e^{\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G} \cdot \frac{\delta}{\delta\varphi}}$ for suitable \mathcal{G} . This is well defined for \mathcal{G} positive but such a transformation is not in general invertible. At an IR fixed point $\mathcal{S}_*[\varphi] \rightarrow T_*[\varphi]$ with T_* obeying a simple linear equation.

The essential observables which appear in this framework are the critical exponents λ^1 which determine how the critical point is approached as $t \rightarrow \infty$ and which are associated with scaling operators \mathcal{O} . The number of negative λ determine the extent to which $\mathcal{S}_0[\varphi]$ has to be tuned so as to lie on the critical surface where the RG flow attains the IR fixed point. The RG equations determine the exponents λ as the eigenvalues of a functional differential operator depending on \mathcal{S}_* . A subspace of the space of operators $\{\mathcal{O}\}$ are redundant operators corresponding to redefinitions $\varphi(x) \rightarrow \psi(x)$ where ψ is a functional of φ . The marginal or zero mode operator \mathcal{Z} is a redundant operator determined by $\psi_{\mathcal{Z}}$ related to a constant rescaling, or reparameterisation, of φ .

In our treatment it is natural to consider also local scaling operators $\Phi_{\Delta}(x)$, functionals of φ , whose scaling dimensions $\Delta \geq 0$ are determined by a local functional eigenvalue equation. For $\Delta = 0$ then $\Phi_0 = 1$, the identity. Any Φ_{Δ} determines an associated \mathcal{O} with $\lambda = \Delta - d$. A particular role in our discussion is played by two exact scaling operators Φ_{δ} and $\Phi_{d-\delta}$ where $\delta = \frac{1}{2}(d - 2 + \eta)$. These both have the form

$$\Phi_{\Delta} = X \cdot \varphi + Y \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_*, \quad (1.1)$$

for appropriate linear operators X, Y . Φ_{δ} gives directly $\psi_{\mathcal{Z}}$. Requiring $\Phi_{\delta} = 0$, for arbitrary η , clearly determines a quadratic form for \mathcal{S}_* and corresponds to the high temperature fixed point. In this case $\mathcal{Z} = 0$ reflecting the fact that η is undetermined in this case. For $\eta = 0$, when $\delta = \delta_0 = \frac{1}{2}(d - 2)$, imposing $\partial^2 \Phi_{\delta_0} \propto \Phi_{d-\delta_0}$ also determines \mathcal{S}_* to be quadratic in φ , corresponding to a free or Gaussian theory. Such a quadratic form depends on a parameter defining a line of equivalent IR fixed points which are generated by \mathcal{Z} . An argument that $\eta = 0$ implies a Gaussian theory in the context of exact RG equations has also been given by Rosten [11].

We also discuss how solutions of the RG flow equations for $\mathcal{S}_t[\varphi]$ can be extended without essential modification to include a source J coupled to φ . This allows the standard vacuum functional $W[J]$ to be directly related to $T[\varphi]$. The connection of the Wilson/Polchinski equation for \mathcal{S}_t with the Wetterich equation for the RG flow of the one particle irreducible functional Γ_t by a Legendre transform is also reviewed. For η non zero it is still feasible to relate the two equations although constructing a Legendre transform with the desired properties involves solving some first order differential equations. The precise form of the

¹Here we choose λ to have the opposite sign to the standard conventions in discussions of critical phenomena.

zero mode operator for the Wetterich equation is constructed by considering the transform of \mathcal{Z} . Various aspects of the discussion are illustrated by consideration of the Gaussian case. This has a limit as $t \rightarrow \infty$ defining a massless theory only when $\eta = 0$, for $\eta \neq 0$ the Gaussian theory approaches the high temperature fixed point. In either case the scaling operators are explicitly constructed.

Although not directly related to our main discussion we also reconsider more briefly the derivative expansion in section 3. This is based on a modification of the Polchinski equation which is tantamount to expanding $\mathcal{S}_t[\varphi]$ in terms of a normal ordered basis of monomials in φ . This allows a derivation of equations, considered by us earlier [31], to first order in the derivative expansion which maintains those exact results for the full RG equations described earlier, namely the existence of eigenfunctions corresponding to scaling dimensions $\frac{1}{2}(d \mp 2 \pm \eta)$ and also a zero mode eigenfunction related to reparameterisation invariance. In section 4 this approach is extended to a supersymmetric theory in three dimensions with $\mathcal{N} = 2$ supersymmetry. In this example η is determined to be $\frac{1}{3}$ for there to be a non trivial IR fixed point.

In appendix A we also outline extensions of part of our discussion to more general RG flow equations without making the restrictions to obtain the Wilson/Polchinski form. In appendix B we also describe in a perturbative context how the zero mode operator is also present.

2 Derivation and Analysis of Exact RG Equations

There are many varieties of essentially equivalent RG flow equations [1, 2, 3, 4]. Assuming for simplicity just a single scalar field $\phi(x) \in V_\phi$, for $x \in \mathbb{R}^d$, they arise by considering regularised actions $\hat{S}_\Lambda[\phi]$ depending on a cut off scale Λ and understanding how these should evolve as Λ varies so as to ensure results independent of the precise form of the cut off may be obtained. Requiring

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-\hat{S}_\Lambda[\phi]} = \frac{\delta}{\delta \phi} \cdot \left(\hat{\Psi}_\Lambda[\phi] e^{-\hat{S}_\Lambda[\phi]} \right), \quad (2.1)$$

for any $\hat{\Psi}_\Lambda[\phi; x] \in V_\phi$, ensures that the basic functional integral defining the quantum field theory for the action \hat{S}_Λ is invariant under changes in Λ . Although the initial choice of \hat{S}_Λ for some $\Lambda = \Lambda_0$ is largely arbitrary there are nevertheless features of the RG flow that are independent of the precise form of \hat{S}_{Λ_0} , at least for appropriate $\hat{\Psi}_\Lambda$ and in suitable limits, which gives rise to the crucial notion of universality. (2.1) is of course equivalent to

$$-\Lambda \frac{\partial}{\partial \Lambda} \hat{S}_\Lambda[\phi] = \hat{\Psi}_\Lambda[\phi] \cdot \frac{\delta}{\delta \phi} \hat{S}_\Lambda[\phi] - \frac{\delta}{\delta \phi} \cdot \hat{\Psi}_\Lambda[\phi]. \quad (2.2)$$

In (2.1) and (2.2)

$$\phi \cdot \psi = \psi \cdot \phi = \int d^d x \phi(x) \psi(x), \quad (2.3)$$

for $\phi, \psi \in V_\phi$. Functional derivatives are here defined so that $\delta F[\phi] = \delta\phi \cdot \frac{\delta}{\delta\phi} F[\phi]$. We also define for $I(x, y) \in V_\phi \times V_\phi$

$$\phi \cdot I \cdot \psi = \psi \cdot I^T \cdot \phi = \int d^d x d^d y \phi(x) I(x, y) \psi(y), \quad (2.4)$$

defining $I : \mathcal{V}_\phi \rightarrow \mathcal{V}_\phi$ with a functional trace

$$\text{tr}(I) = \int d^d x I(x, x). \quad (2.5)$$

As is commonplace it is natural to transform the fields to momentum space, $\phi(x) \rightarrow \tilde{\phi}(p)$, so that

$$\phi \cdot \psi = \frac{1}{(2\pi)^d} \int d^d p \tilde{\phi}(p) \tilde{\psi}(-p). \quad (2.6)$$

For a translation invariant I in (2.4), so that $I(x, y) \rightarrow G(x - y)$, then in momentum space $\tilde{I}(p, q) \rightarrow \tilde{G}(q) (2\pi)^d \delta^d(p + q)$,

$$\phi \cdot G \cdot \psi = \frac{1}{(2\pi)^d} \int d^d p \tilde{\phi}(p) \tilde{G}(p) \tilde{\psi}(-p), \quad (2.7)$$

and of course $\widetilde{G_1 \cdot G_2}(p) = \tilde{G}_1(p) \tilde{G}_2(p)$ and $\tilde{G}^T(p) = \tilde{G}(-p)$. Furthermore in this case

$$\text{tr}(G) = V \frac{1}{(2\pi)^d} \int d^d p \tilde{G}(p), \quad (2.8)$$

where V is the spatial volume, implicitly assuming a spatial cut off, such as compactifying on a torus, but whose details are unspecified here.

The cut off Λ sets the fundamental scale and the equations are further assumed to be reduced to dimensionless form by requiring

$$\phi(x) = \Lambda^{\delta_0} \varphi(x\Lambda), \quad \frac{\delta}{\delta\phi(x)} = \Lambda^{d-\delta_0} \frac{\delta}{\delta\varphi(x\Lambda)} \quad \hat{\Psi}_\Lambda(x) = \Lambda^{\delta_0} \Psi_t(x\Lambda), \quad (2.9)$$

for some choice of δ_0 , with

$$t = -\ln \Lambda/\Lambda_0, \quad \hat{S}_\Lambda[\phi] = S_t[\varphi]. \quad (2.10)$$

With the rescaling (2.9), (2.2) takes the form

$$\left(\frac{\partial}{\partial t} + D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta\varphi} - dV \frac{\partial}{\partial V} \right) S_t[\varphi] = \Psi_t[\varphi] \cdot \frac{\delta}{\delta\varphi} S_t[\varphi] - \frac{\delta}{\delta\varphi} \cdot \Psi_t[\varphi], \quad (2.11)$$

for $D^{(\delta)}$ defined by

$$D^{(\delta)} \varphi(x) = (x \cdot \partial_x + \delta) \varphi(x), \quad D^{(\delta)} \tilde{\varphi}(p) = -(p \cdot \partial_p + d - \delta) \tilde{\varphi}(p), \quad (2.12)$$

and

$$D^{(\delta)} \varphi \cdot \frac{\delta}{\delta\varphi} = \int d^d x D^{(\delta)} \varphi(x) \frac{\delta}{\delta\varphi(x)} = \frac{1}{(2\pi)^d} \int d^d p D^{(\delta)} \tilde{\varphi}(p) \frac{\delta}{\delta\tilde{\varphi}(p)} \quad (2.13)$$

Subsequently we frequently use

$$D^{(\delta)}\varphi \cdot \psi = -\varphi \cdot D^{(d-\delta)}\psi = \varphi \overleftarrow{D}^{(\delta)} \cdot \psi = -\psi \overleftarrow{D}^{(d-\delta)} \cdot \varphi. \quad (2.14)$$

For a well defined RG equation Ψ_t , or $\hat{\Psi}_\Lambda$, should be such that $S_t[\varphi] \in \mathcal{M}$ under evolution in t for all finite $t > 0$, so that $S_t[\varphi]$ is always essentially local, as required for a Wilsonian action. In general the resulting equations should also be in accord with the expected irreversibility of RG flow towards IR fixed points so that the detailed form for the initial S_0 becomes largely irrelevant when $t \rightarrow \infty$. Contributions to $S_t[\varphi]$ proportional to the volume V and independent of φ are irrelevant for most purposes. It is consistent to restrict to equivalence classes defined by $S_t[\varphi] \sim S_t[\varphi] + c_t V$ which justifies the common neglect of contributions involving V . However such terms are generated by the RG flow in general so it is therefore necessary to include V in the basis \mathcal{M} if this is to be closed under RG flow. The V -dependent terms in (2.11), which of course reflect the variation in the overall volume due to the rescaling of x in (2.9), are also necessary to ensure a consistent derivative expansion when the leading term in $S_t[\varphi]$ assumes constant φ .

Although the class of possible Ψ_t satisfying the above conditions is not clear cut (a more general discussion is given in appendix A) we focus here on the choice due to Wilson which corresponds to taking

$$\hat{\Psi}_\Lambda[\phi] = \frac{1}{2} \hat{G}_\Lambda \cdot \frac{\delta}{\delta \phi} \hat{S}_\Lambda[\phi] - \hat{H}_\Lambda \cdot \phi, \quad (2.15)$$

or, imposing agreement with (2.9),²

$$\hat{G}_\Lambda(y) = \Lambda^{2\delta_0} G(y\Lambda), \quad \hat{H}_\Lambda(y) = \Lambda^d H(y\Lambda), \quad (2.16)$$

then

$$\Psi_t[\varphi] = \frac{1}{2} G \cdot \frac{\delta}{\delta \varphi} S_t[\varphi] - H \cdot \varphi. \quad (2.17)$$

In this case (2.11) becomes

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + (D^{(\delta_0)}\varphi + H \cdot \varphi) \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) S_t[\varphi] \\ &= \frac{1}{2} \frac{\delta}{\delta \varphi} S_t[\varphi] \cdot G \cdot \frac{\delta}{\delta \varphi} S_t[\varphi] - \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi} S_t[\varphi] + \text{tr}(H). \end{aligned} \quad (2.18)$$

Clearly it is sufficient to assume G is symmetric, $G = G^T$. Extending standard results for partial different equations (2.18) is a well defined parabolic functional differential equation for S_t so long as G is negative definite and is then soluble for S_t in terms of S_0 for $t > 0$. The existence of solutions for general initial S_0 only for t increasing reflects the irreversibility of RG flow. The locality requirements necessitate that $\tilde{G}(p)$ and $\tilde{H}(p)$ should be analytic in p in the neighbourhood of $p = 0$.

Apart from the $\text{tr}(H)$ term the equation (2.18) is invariant under

$$\tilde{\varphi}(p) \rightarrow e^{h(p)} \tilde{\varphi}(p), \quad \tilde{G}(p) \rightarrow e^{h(p)+h(-p)} \tilde{G}(p), \quad \tilde{H}(p) \rightarrow \tilde{H}(p) + p \cdot \partial_p h(p). \quad (2.19)$$

²In the original proposal the restriction $\hat{H}_\Lambda = -\hat{G}_\Lambda$ was also made, requiring by virtue of (2.16) $\delta_0 = \frac{1}{2}d$, but this appears unnecessary.

For consistency with locality $h(p)$ should also be required to be analytic in p for $p \approx 0$. The freedom in (2.19) shows that $\tilde{H}(p)$ may be chosen at will save for $\tilde{H}(0)$ although in $D^{(\delta_0)}\varphi + H \cdot \varphi$ only $\delta_0 + \tilde{H}(0)$ has significance.

A variant RG equation due to Polchinski [4] is obtained by writing

$$\mathcal{S}_t[\varphi] = \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \frac{1}{2d} \text{tr}(\mathcal{G} \cdot \mathcal{G}^{-1}) + \mathcal{S}_t[\varphi], \quad (2.20)$$

for \mathcal{G} symmetric and with $\tilde{\mathcal{G}}(p)$ chosen in due course as the regularised propagator ensuring a finite functional integral. Substituting in (2.18) gives

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + (D^{(\delta_0)}\varphi - G \cdot \mathcal{G}^{-1} \cdot \varphi + H \cdot \varphi) \cdot \frac{\delta}{\delta\varphi} - dV \frac{\partial}{\partial V} \right) \mathcal{S}_t[\varphi] \\ &= \frac{1}{2} \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] \\ & \quad - (D^{(\delta_0)}\varphi + H \cdot \varphi) \cdot \mathcal{G}^{-1} \cdot \varphi + \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1} \cdot \varphi + \text{tr}(H - G \cdot \mathcal{G}^{-1}), \end{aligned} \quad (2.21)$$

Choosing now

$$H = G \cdot \mathcal{G}^{-1} + \frac{1}{2}\eta \mathbf{1}, \quad (2.22)$$

and also requiring

$$D^{(d-\delta_0)} \mathcal{G}^{-1} + \mathcal{G}^{-1} \overleftarrow{D}^{(d-\delta_0)} = \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1}, \quad (2.23)$$

(2.21) becomes the Polchinski RG equation [4], extended to include the parameter η [13],

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} - dV \frac{\partial}{\partial V} \right) \mathcal{S}_t[\varphi] \\ &= \frac{1}{2} \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] - \frac{1}{2} \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(\mathbf{1})), \end{aligned} \quad (2.24)$$

for a modified scaling dimension

$$\delta = \delta_0 + \frac{1}{2}\eta. \quad (2.25)$$

Since $\mathcal{G}^{-1} \cdot (D^{(d-\delta_0)} \mathcal{G}^{-1} + \mathcal{G}^{-1} \overleftarrow{D}^{(d-\delta_0)}) \cdot \mathcal{G}^{-1} = -D^{(\delta_0)} \mathcal{G} - \mathcal{G} \overleftarrow{D}^{(\delta_0)}$ from (2.23)

$$D^{(\delta_0)} \mathcal{G} + \mathcal{G} \overleftarrow{D}^{(\delta_0)} = -G, \quad (2.26)$$

which implies, for translation invariant \mathcal{G} ,

$$(x \cdot \partial_x + 2\delta_0) \mathcal{G}(x) = -G(x), \quad (p \cdot \partial_p + d - 2\delta_0) \tilde{\mathcal{G}}(p) = \tilde{G}(p). \quad (2.27)$$

In order to ensure that IR behaviour is not modified by the introduction of a cut off the regularised propagator \mathcal{G} must have the same long distance behaviour as for a free field, so that δ has then to be identified with the canonical dimension of the scalar field, giving

$$\delta_0 = \frac{1}{2}(d-2), \quad (2.28)$$

and with standard conventions this requires

$$\tilde{\mathcal{G}}(p) = \frac{K(p^2)}{p^2}, \quad K(0) = 1, \quad K(p^2) \xrightarrow{p^2 \rightarrow \infty} 0, \quad (2.29)$$

for $K(p^2)$ a smooth cut off function, analytic in p^2 in the neighbourhood of $p^2 = 0$. The expression given by (2.29) for \mathcal{G} , (2.27) then determines G in terms of the cut off K

$$\tilde{G}(p) = 2K'(p^2). \quad (2.30)$$

Alternatively

$$\tilde{\mathcal{G}}_\Lambda(p) = \frac{K(p^2/\Lambda^2)}{p^2} \Rightarrow \dot{\mathcal{G}} \equiv \frac{\partial}{\partial t} \mathcal{G}_\Lambda \Big|_{\Lambda=1} = G. \quad (2.31)$$

These results ensure that $\text{tr}(G \cdot \mathcal{G}^{-1})/2d$ is equal, neglecting divergent surface terms arising in integration by parts, to $\frac{1}{2} \text{tr} \ln(\mathcal{G}^{-1})$ so this contribution in (2.20) just cancels the one loop functional determinant arising from the functional integration of $\frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \varphi$.

In terms of the cut off function K the Polchinski RG equation (2.24) may then be written explicitly as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) \mathcal{S}_t &= \frac{1}{(2\pi)^d} \int d^d p K'(p^2) \left(\frac{\delta \mathcal{S}_t}{\delta \tilde{\varphi}(p)} \frac{\delta \mathcal{S}_t}{\delta \tilde{\varphi}(-p)} - \frac{\delta^2 \mathcal{S}_t}{\delta \tilde{\varphi}(p) \delta \tilde{\varphi}(-p)} \right) \\ &\quad - \frac{\eta}{2(2\pi)^d} \int d^d p \left(\tilde{\varphi}(p) \frac{p^2}{K(p^2)} \tilde{\varphi}(-p) - V \right). \end{aligned} \quad (2.32)$$

For this parabolic equation to be well defined for increasing t we must have

$$K'(p^2) < 0, \quad (2.33)$$

which is compatible with (2.29). By combining (2.25) with (2.28) we have

$$\delta = \frac{1}{2}(d - 2 + \eta). \quad (2.34)$$

For $\eta = 0$ the Polchinski equation (2.24) is of course identical with the Wilson equation (2.21) if we set $H = 0$ although there is now, by virtue of (2.20), a Gaussian solution for $\mathcal{S}_t = 0$.

In each of the RG flow equations the coefficient functions, G, H in (2.18) or G, \mathcal{G}^{-1} in (2.24), have no short distance singularities, as a consequence of the analyticity assumptions for $\tilde{G}(p), \tilde{H}(p)$, so that S_t or \mathcal{S}_t should not contain any such singularities under t evolution for finite t . In this sense the RG flow preserves locality.

2.1 Functional Space

For non linear differential equations it is necessary to specify the class of functions and associated boundary conditions which form the appropriate solution space. The functional differential RG equations described here are assumed to act on an infinite dimensional vector space of action functionals \mathcal{M} spanned by monomials in φ

$$\{ V, \mathcal{P}_n[\varphi], n = 1, 2, \dots \}, \quad \mathcal{P}_n[\varphi] = \int \prod_{r=1}^n d^d x_r \varphi(x_r) G_n(x_1, \dots, x_n), \quad (2.35)$$

where $\{\mathcal{P}_n[\varphi]\}$ are defined in terms of symmetric functions $\{G_n(x_1, \dots, x_n)\}$ which are restricted so that each G_n is translation invariant

$$G_n(x_1 + a, \dots, x_n + a) = G_n(x_1, \dots, x_n), \quad (2.36)$$

and also connected requiring

$$G_n(x_{1,a}, \dots, x_{n,a}) \rightarrow 0 \quad \text{as} \quad a \rightarrow \infty \quad x_{r,a} = \begin{cases} x_r + a, & r \in S \subset \{1, \dots, n\} \\ x_r, & r \in \{1, \dots, n\} \setminus S \end{cases}, \quad (2.37)$$

for all proper subsets S . Furthermore the cut off should ensure that all G_n are quasi-local in that they are non singular at short distances, $G_n(x_1, \dots, x_n)$ is regular for any $x_r - x_s \rightarrow 0$, $r \neq s$. Equivalently

$$\int \prod_{r=1,n} d^d x_r e^{i p_r \cdot x_r} G_n(x_1, \dots, x_n) = (2\pi)^d \delta^d(\sum_{r=1,n} p_r) \tilde{G}_n(p_1, \dots, p_n), \quad (2.38)$$

where $\tilde{G}_n(p_1, \dots, p_n)$, restricted to $\sum_{r=1,n} p_r = 0$, is a symmetric function analytic in each p_r in the neighbourhood of $p_r = 0$. By expanding $\tilde{G}_n(p_1, \dots, p_n)$ about $p_r = 0$ we may consider a basis of local operators $\mathcal{M}_{\text{local}}$ where G_n in (2.35) is restricted so that $G_n(x_1, \dots, x_n) = 0$ if $x_{rs} \neq 0$, for all $r \neq s$, or $\tilde{G}_n(p_1, \dots, p_n)$ is just a polynomial in each p_r . The local operators then have the form

$$\mathcal{P}_n[\varphi] = \int d^d x P_{n,s}[\varphi; x], \quad (2.39)$$

for

$$P_{n,s}[\varphi; x] = \prod \partial^{s_i} \varphi(x) = O(\varphi^n, \partial^s), \quad s = \sum_i s_i, \quad (2.40)$$

depending only on $\varphi(x)$ and its derivatives, all indices contracted to form a scalar. Of course such local operators (2.40) may be extended to include also operators with non zero spin.

For functionals of finite order in φ as in (2.35) then an alternative normal ordered form, relative to a two point function \mathcal{G} , is defined by

$$\mathcal{N}_{\mathcal{G}}(\mathcal{P}_n[\varphi]) = e^{-\frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}} \mathcal{P}_n[\varphi]. \quad (2.41)$$

In perturbative expansions, with a propagator \mathcal{G} , this removes contractions between different φ in $\mathcal{P}_n[\varphi]$. In (2.24) we may write $\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1) = \mathcal{N}_{\mathcal{G}}(\varphi \cdot \mathcal{G}^{-1} \cdot \varphi)$. Although the $\text{tr}(1)$ term involves a divergent p -integration, as exhibited in (2.32), its subtraction ensures contributions arising from $\varphi \cdot \mathcal{G}^{-1} \cdot \varphi$ are well defined in later manipulations.

2.2 Linearisation of RG Equations

The exact RG equations are non linear but as shown by Rosten [10], and also in [14], they may be linearised and the RG flow is then determined by standard β -functions. Defining

$$\mathcal{Y} = \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}, \quad (2.42)$$

which may be expressed in an analogous fashion to (2.4), then, using (2.26) with δ_0 given by (2.28),

$$\left[D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta \varphi}, \mathcal{Y} \right] = \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi}, \quad (2.43)$$

so that

$$D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta \varphi} e^{\mathcal{Y}} = e^{\mathcal{Y}} \left(D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi} \right). \quad (2.44)$$

Hence from (2.24) for $\eta = 0$,

$$\left(\frac{\partial}{\partial t} + D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) \left(e^{\mathcal{Y}} e^{-S_t[\varphi]} \right) = 0. \quad (2.45)$$

So long as \mathcal{G} is positive definite, requiring $K(p^2) > 0$, $e^{\mathcal{Y}}$ has a well defined action but is not invertible for any general $S_t[\varphi]$. For $t = 0$

$$e^{\mathcal{Y}} e^{-S_0[\varphi]} = e^{T[\varphi] + cV}, \quad (2.46)$$

where $T[\varphi] + cV$, $\partial_V T[\varphi] = 0$, $\delta_\varphi c = 0$, may be evaluated by a perturbative expansion in terms of the contributions for all connected vacuum Feynman graphs with propagators given by $\tilde{\mathcal{G}}(p)$, as in (2.29) and which are singular at $p^2 = 0$, and vertices determined by $S_0[\varphi]$ which may be assumed to be a conventional action formed by a finite rotationally invariant polynomial in the scalar field φ and its derivatives. As shown later $T[\varphi]$ is equivalent to the normal vacuum functional $W[J]$ and is inherently non local. In terms of $T[\varphi]$ the linear equation (2.45) for the t dependence is easily solved giving

$$e^{\mathcal{Y}} e^{-S_t[\varphi]} = e^{T[\varphi_t] + c e^{dt} V}, \quad (2.47)$$

for

$$\varphi_t(x) = e^{-\frac{1}{2}(d-2)t} \varphi(e^{-t}x) \quad \text{or} \quad \tilde{\varphi}_t(p) = e^{\frac{1}{2}(d+2)t} \tilde{\varphi}(e^t p). \quad (2.48)$$

In terms of the original field ϕ before rescaling $\varphi_t(x) = \Lambda_0^{-\frac{1}{2}(d-2)} \phi(x/\Lambda_0)$.

(2.47) and (2.46) may be combined to give

$$e^{\mathcal{Y}} e^{-S_t[\varphi]} = e^{\mathcal{Y}_t} e^{-S_0[\varphi_t]} e^{c(e^{dt}-1)V}, \quad (2.49)$$

where

$$\mathcal{Y}_t = \frac{1}{2} \frac{\delta}{\delta \varphi_t} \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi_t}, \quad \frac{\delta}{\delta \varphi_t(x)} = e^{-\frac{1}{2}(d+2)t} \frac{\delta}{\delta \varphi(e^{-t}x)}. \quad (2.50)$$

(2.49) may be solved for S_t in the form [15, 16, 17]

$$e^{-S_t[\varphi]} = e^{\mathcal{Y}_t - \mathcal{Y}} e^{-S_0[\varphi_t]} e^{c(e^{dt}-1)V}, \quad (2.51)$$

since

$$\mathcal{Y}_t - \mathcal{Y} = \frac{1}{2} \frac{\delta}{\delta \varphi_t} \cdot \mathcal{X}_t \cdot \frac{\delta}{\delta \varphi_t}, \quad \tilde{\mathcal{X}}_t(p) = \frac{1}{p^2} (K(e^{-2t}p^2) - K(p^2)) > 0 \quad \text{for } t > 0, \quad (2.52)$$

as a consequence of (2.33). Hence $e^{\mathcal{Y}_t - \mathcal{Y}}$ is well defined so long as $t \geq 0$. The solution verifies that S_t does not have singularities for $p \approx 0$ for suitably local S_0 .

Such linearisation allows a direct connection with standard linear RG flow equations involving β -functions. In general a RG flow $\mathcal{S}_t[\varphi]$ determines a trajectory in an infinite dimensional space of all possible $\{\mathcal{S}[\varphi]\}$ actions consistent with the symmetries of the initial $\mathcal{S}_0[\varphi]$. Coordinates on this space may be identified with an infinite set of couplings $\{g\}$ so that from (2.47)

$$\mathcal{S}_t[\varphi] = \mathcal{S}[\varphi; g_t] \quad \Rightarrow \quad T[\varphi_t] = T[\varphi; g_t]. \quad (2.53)$$

Clearly $T[\varphi; g] = T[\varphi_{-t}; g_t]$. Defining the β -functions as usual as the tangent vectors to the RG flow of the couplings g_t ,

$$\frac{d}{dt} g_t = \beta(g_t), \quad (2.54)$$

then (2.45) and (2.53) require

$$\left(D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} + \beta(g) \cdot \frac{\partial}{\partial g} \right) T[\varphi; g] = 0, \quad (2.55)$$

which is a standard linear RG equation for $\beta(g) \cdot \frac{\partial}{\partial g} = \beta^i(g) \frac{\partial}{\partial g^i}$.

If instead we assume the \mathcal{S}_t evolves according to the Polchinski equation (2.24) with $\eta \neq 0$ then using

$$e^{\mathcal{Y}} \left(\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1) - \varphi \cdot \frac{\delta}{\delta \varphi} \right) = \left(\varphi \cdot \mathcal{G}^{-1} \cdot \varphi + \varphi \cdot \frac{\delta}{\delta \varphi} \right) e^{\mathcal{Y}}, \quad (2.56)$$

which follows from

$$\left[\mathcal{Y}, \varphi \cdot \frac{\delta}{\delta \varphi} \right] = 2\mathcal{Y}, \quad \left[\mathcal{Y}, \varphi \cdot \mathcal{G}^{-1} \cdot \varphi \right] = 2\varphi \cdot \frac{\delta}{\delta \varphi} + \text{tr}(1), \quad (2.57)$$

(2.45) is replaced by

$$\left(\frac{\partial}{\partial t} + D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} - \frac{1}{2} \eta \varphi \cdot \mathcal{G}^{-1} \cdot \varphi \right) \left(e^{\mathcal{Y}} e^{-\mathcal{S}_t[\varphi]} \right) = 0. \quad (2.58)$$

The solution (2.47) becomes instead

$$e^{\mathcal{Y}} e^{-\mathcal{S}_t[\varphi]} = e^{T[\varphi_t] + \frac{1}{2} \varphi_t \cdot h \cdot \varphi_t - \frac{1}{2} \varphi \cdot h \cdot \varphi + c e^{dt} V}, \quad (2.59)$$

where now

$$\varphi_t(x) = e^{-\frac{1}{2}(d-2-\eta)t} \varphi(e^{-t}x), \quad (2.60)$$

and we require

$$D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} (\varphi \cdot h \cdot \varphi) = -\eta \varphi \cdot \mathcal{G}^{-1} \cdot \varphi, \quad (2.61)$$

which is equivalent to

$$D^{(d-\delta+\eta)} h + h \overleftarrow{D}^{(d-\delta+\eta)} = \eta \mathcal{G}^{-1}. \quad (2.62)$$

In (2.59) we have imposed the initial condition that the vacuum functional $T[\varphi]$ is given by $e^{\mathcal{Y}} e^{-\mathcal{S}_0[\varphi]}$ and so is identical with T in (2.46). Writing $\varphi \cdot h \cdot \varphi$ in the form given by (2.7) then (2.62) requires

$$\left(1 + \frac{1}{2} \eta - p^2 \frac{\partial}{\partial p^2} \right) \tilde{h}(p) = \frac{1}{2} \eta \frac{p^2}{K(p^2)}. \quad (2.63)$$

This has a solution, so long as $\eta < 2$,

$$\tilde{h}(p) = \frac{p^2}{K(p^2)}(1 - \sigma_*(p^2)), \quad \sigma_*(p^2) = K(p^2)(p^2)^{\frac{1}{2}\eta} \int_0^{p^2} dx x^{-\frac{1}{2}\eta} \frac{d}{dx} \frac{1}{K(x)}, \quad (2.64)$$

where we have imposed the condition that $h(p)$ should be analytic, for general η , when $p \approx 0$ to eliminate solutions of the homogeneous equation proportional to $(p^2)^{1+\frac{1}{2}\eta}$. The expression (2.64) gives for any $\eta < 2$

$$\sigma_*(p^2) \sim -\frac{K'(0)}{1 - \frac{1}{2}\eta} p^2, \quad h(p) \sim p^2 \quad \text{as } p^2 \rightarrow 0, \quad (2.65)$$

As a special case

$$\sigma_*(p^2)|_{\eta=0} = 1 - K(p^2), \quad \tilde{h}(p)|_{\eta=0} = p^2. \quad (2.66)$$

If $\eta < 0$ then it is possible to integrate by parts so as to combine the two terms in the expression for $\tilde{h}(p)$ in (2.64) into a single integral.

For the Polchinski RG equation with $\eta \neq 0$ there is still a flow in the space of couplings as determined by (2.53) but the trajectory g_t is now modified as is therefore the beta function $\beta(g, \eta)$, given by (2.54), which now depends on η . In this case

$$T[\varphi_t; g] + \frac{1}{2} \varphi_t \cdot h \cdot \varphi_t - \frac{1}{2} \varphi \cdot h \cdot \varphi = T[\varphi; g_t], \quad (2.67)$$

and instead of (2.55) we now have

$$\left(D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta\varphi} - dV \frac{\partial}{\partial V} + \beta(g, \eta) \cdot \frac{\partial}{\partial g} \right) T[\varphi; g] = \frac{1}{2} \eta \varphi \cdot \mathcal{G}^{-1} \cdot \varphi, \quad (2.68)$$

although $T[\varphi; g]$ is unchanged. However the apparent dependence on η is superfluous. For a general set of couplings $\{g\}$ rescaling of the field φ may be compensated by a corresponding change in g giving

$$\left(\varphi \cdot \frac{\delta}{\delta\varphi} + \kappa(g) \cdot \frac{\partial}{\partial g} \right) \mathcal{S}[\varphi; g] = -\varphi \cdot \mathcal{G}^{-1} \cdot \varphi, \quad (2.69)$$

for some appropriate $\kappa(g)$. Equivalently $\mathcal{S}[\varphi; g] = \mathcal{S}[e^s \varphi; g_s] + \frac{1}{2}(e^{2s} - 1) \varphi \cdot \mathcal{G}^{-1} \cdot \varphi$ where $\frac{d}{ds} g_s = \kappa(g_s)$. Using (2.56) this ensures

$$\left(\varphi \cdot \frac{\delta}{\delta\varphi} - \kappa(g) \cdot \frac{\partial}{\partial g} \right) T[\varphi; g] = -\varphi \cdot \mathcal{G}^{-1} \cdot \varphi, \quad (2.70)$$

Applying (2.70) reduces (2.68) to (2.55) so long as

$$\beta(g, \eta) = \beta(g) - \frac{1}{2} \eta \kappa(g). \quad (2.71)$$

2.3 IR Fixed Points

In the above discussion η is undetermined and may be chosen at will. The crucial constraint, which provides a determination of η , is that there should exist a well defined non zero non trivial limit as $t \rightarrow \infty$,

$$\mathcal{S}_t[\varphi] \xrightarrow{t \rightarrow \infty} \mathcal{S}_*[\varphi] = \mathcal{S}[\varphi; g_*]. \quad (2.72)$$

where from (2.24) we must have

$$\begin{aligned} E[\varphi] &\equiv D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_*[\varphi] - dV \frac{\partial}{\partial V}\mathcal{S}_*[\varphi] - \frac{1}{2} \frac{\delta}{\delta\varphi}\mathcal{S}_*[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_*[\varphi] + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_*[\varphi] \\ &= -\frac{1}{2}\eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1)), \end{aligned} \quad (2.73)$$

with δ, η related by (2.34). In general non trivial IR fixed points, with long range order, are only possible if $\mathcal{S}_0[\varphi]$ is restricted to belong to a critical surface of codimension equal to $N_{\text{rel}} > 0$, the number of relevant operators at the particular fixed point. There may of course be more than one IR fixed point, with differing possible N_{rel} and η , which may be obtained as $t \rightarrow \infty$ depending on the precise initial $\mathcal{S}_0[\varphi]$.

In (2.59) then, with φ_t given by (2.60), it is easy to verify that

$$\varphi_t \cdot h \cdot \varphi_t \underset{t \rightarrow \infty}{\sim} e^{\eta t} \varphi \cdot \mathcal{G}_0^{-1} \cdot \varphi \quad \varphi \cdot \mathcal{G}_0^{-1} \cdot \varphi = \frac{1}{(2\pi)^d} \int d^d p \, \varphi(p) p^2 \varphi(-p), \quad (2.74)$$

where

$$\mathcal{G}_0(y) = \frac{k}{(y^2)^{\frac{1}{2}(d-2)}}, \quad \tilde{\mathcal{G}}_0(p) = \frac{1}{p^2}, \quad (2.75)$$

is just the Green function for $-\partial^2$, which is independent of the cut off and so satisfies, instead of (2.26),

$$D^{(\delta_0)} \mathcal{G}_0 + \mathcal{G}_0 \overleftarrow{D}^{(\delta_0)} = 0. \quad (2.76)$$

In order to ensure that (2.59) is compatible with (2.72) it is necessary to require

$$T[\varphi_t] \underset{t \rightarrow \infty}{\sim} -e^{\eta t} \frac{1}{2} \varphi \cdot \mathcal{G}_0^{-1} \cdot \varphi + T_*[\varphi]. \quad (2.77)$$

Except for the local term proportional to $e^{\eta t}$, which diverges for $\eta > 0$, (2.77) shows that $T[\varphi_t]$ has well defined limit as $t \rightarrow \infty$ if there is a non trivial IR fixed point. Such a limit can only be possible for one precise value of η which, as demonstrated later, can be identified with the anomalous scale dimension of φ at the fixed point. For a non trivial limit in (2.77) T_* must scale invariant satisfying $T_*[\varphi_t] = T_*[\varphi]$ and from (2.72), (2.77) we then have

$$e^{T_*[\varphi] - \frac{1}{2} \varphi \cdot h \cdot \varphi} = e^{\mathcal{Y}} e^{-\mathcal{S}_*[\varphi]}, \quad (2.78)$$

and $T[\varphi; g_*] = T_*[\varphi] - \frac{1}{2} \varphi \cdot h \cdot \varphi$. Assuming

$$\beta(g_*, \eta) = 0, \quad (2.79)$$

with η determined by the existence of a non trivial limit, then using (2.68) with (2.61) ensures that

$$\left(D^{(\delta-\eta)}\varphi \cdot \frac{\delta}{\delta\varphi} - dV \frac{\partial}{\partial V} \right) T_*[\varphi] = 0. \quad (2.80)$$

As the fixed point is approached it is standard to assume

$$\mathcal{S}_t[\varphi] \sim \mathcal{S}_*[\varphi] - \sum_{n \geq 0} e^{-\lambda_n t} \mathcal{O}_n[\varphi] \quad \text{as } t \rightarrow \infty, \quad (2.81)$$

for $\{\mathcal{O}_n\}$ the set of all scaling operators and $\lambda_n = \Delta_n - d$, where Δ_n is the scale dimension of an associated local operator, determine the critical exponents. These may be ordered so that $\lambda_{n+1} > \lambda_n$ with $\lambda_0 = -d$ and $\mathcal{O}_0 \propto V$ corresponding to the identity operator, which is independent of φ . In general $\lambda_1 = -\frac{1}{2}(d + 2 - \eta)$ when \mathcal{O}_1 is essentially just φ itself. For $\lambda_n < 0$, which defines relevant operators, then \mathcal{O}_n , $n = 0, 1, \dots, N_{\text{rel}} - 1$, in general corresponds to a composite operator constructed from φ^n . If there is a \mathbb{Z}_2 symmetry under $\varphi \leftrightarrow -\varphi$ so that $\mathcal{S}_0[\varphi] = \mathcal{S}_0[-\varphi]$ then the sum in (2.81) may be restricted to even n corresponding to operators $\mathcal{O}_n[\varphi] = \mathcal{O}_n[-\varphi]$. The critical surface corresponds to no contributions in (2.81) with $\lambda_n < 0$ so that (2.72) is well defined. This provided N_{rel} conditions on $\mathcal{S}_0[\varphi]$, or equivalently the initial g . Under RG flow the couplings g_t are therefore restricted to a critical subspace of codimension N_{rel} in the total space of couplings.

In conjunction with (2.47) (2.81) would require $T[\varphi_t]$ to have contributions $\propto e^{-\lambda_n t}$ as $t \rightarrow \infty$ where in general the differing $\{\lambda_n\}$ are not commensurate. Such terms may correspond to the divergences present at coincident points reflecting the singular coefficient functions in the operator product expansion.

2.4 Eigenvalue Equations

The determination of the exponents λ_n in (2.81) is associated with an eigenvalue problem

$$\Delta_{\mathcal{S}*} \mathcal{O} = \lambda \mathcal{O}, \quad \mathcal{O} \in \mathcal{M}, \quad (2.82)$$

where $\Delta_{\mathcal{S}}$ is the functional differential operator depending on action functionals $S[\varphi] \in \mathcal{M}$

$$\Delta_{\mathcal{S}} = D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} - \frac{\delta}{\delta \varphi} S[\varphi] \cdot G \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi}, \quad (2.83)$$

and in (2.82) $S \rightarrow \mathcal{S}_*$, the fixed point action determined by (2.73), with δ determined in terms of η by (2.34).

Associated with (2.82) there is a corresponding eigenvalue problem for local operators $\mathcal{M}' = \{\Phi[\varphi; x]\}$, which are functionals of φ depending also on x . \mathcal{M}' is assumed to have a similar basis to (2.35) for \mathcal{M} but $V \rightarrow 1$, the identity operator, and $G_n(x_1, \dots, x_n) \rightarrow G_n(x; x_1, \dots, x_n)$ which are similarly translationally invariant under a translation in x and each x_r , connected, quasi-local and symmetric functions of (x_1, \dots, x_n) . Additionally $\mathcal{M}'_{\text{local}}$ is such that $\Phi(x) \in \mathcal{M}'_{\text{local}}$ is formed just from $\varphi(x)$ and its derivatives as in (2.40). The eigenvalue equation determining the scaling dimension Δ becomes

$$\Delta_{\mathcal{S}*, \text{loc}} \Phi_{\Delta}(x) = D^{(\Delta)} \Phi_{\Delta}(x) = (x \cdot \partial + \Delta) \Phi_{\Delta}(x), \quad (2.84)$$

for $\Delta_{\mathcal{S}*, \text{loc}}$, acting on \mathcal{M}' , given by

$$\Delta_{\mathcal{S}*, \text{loc}} = D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - \frac{\delta}{\delta \varphi} \mathcal{S}_*[\varphi] \cdot G \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi}. \quad (2.85)$$

This is the same operator as in (2.82) but omitting the V dependent terms. Manifestly $\Delta_{\mathcal{S}*, \text{loc}} 1 = 0$. Defining $1 \cdot \Phi = \int d^d x \Phi(x) = \tilde{\Phi}(0)$, so that $1 \cdot 1 = V$, then from (2.84) there is an associated $\mathcal{O} \in \mathcal{M}$ satisfying (2.82) since

$$\Delta_{\mathcal{S}*} (1 \cdot \Phi_{\Delta}) = 1 \cdot \Delta_{\mathcal{S}*, \text{loc}} \Phi_{\Delta} - dV \frac{\partial}{\partial V} (1 \cdot \Phi_{\Delta}) = (\Delta - d) 1 \cdot \Phi_{\Delta}. \quad (2.86)$$

For two cases exact local eigenoperators can be constructed starting from

$$\Delta_{\mathcal{S}_*, \text{loc}} \varphi = D^{(\delta)} \varphi - G \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_*, \quad (2.87)$$

and, by taking the derivative of (2.73),

$$\Delta_{\mathcal{S}_*, \text{loc}} \frac{\delta}{\delta \varphi} \mathcal{S}_* = D^{(d-\delta)} \frac{\delta}{\delta \varphi} \mathcal{S}_* - \eta \mathcal{G}^{-1} \cdot \varphi = 0. \quad (2.88)$$

Assuming the form (1.1) then (2.84) requires

$$\begin{aligned} X \overleftarrow{D}^{(d-\delta)} + D^{(\Delta)} X &= -\eta Y \cdot \mathcal{G}^{-1}, \\ Y \overleftarrow{D}^{(\delta)} + D^{(\Delta)} Y &= X \cdot G, \end{aligned} \quad (2.89)$$

where for locality $\tilde{X}(p), \tilde{Y}(p)$ are analytic for $p \approx 0$. Using (2.26), (2.62) and also (2.76) the solutions of (2.89) give

$$\Phi_\delta = \mathcal{G}_0 \cdot \left(h \cdot \varphi - (1 - h \cdot \mathcal{G}) \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* \right) = \mathcal{G}_0 \cdot \left(\mathcal{G}^{-1} \cdot \varphi - (1 - h \cdot \mathcal{G}) \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* \right), \quad (2.90a)$$

$$\Phi_{d-\delta} = \mathcal{G}_0^{-1} \cdot \left(\varphi + \mathcal{G} \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* \right) = \mathcal{G}_0^{-1} \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_*. \quad (2.90b)$$

The quasi-locality of $\Phi_\delta, \Phi_{d-\delta}$ follows since $\mathcal{G}_0^{-1} \cdot \mathcal{G}, \mathcal{G}_0 \cdot \mathcal{G}^{-1}$ and $\mathcal{G}_0 \cdot (1 - h \cdot \mathcal{G})$ have no singularities at $p = 0$ (from (2.64) and (2.65) $1 - \tilde{\mathcal{G}}(p) \tilde{h}(p) = \sigma_*(p^2) = \mathcal{O}(p^2)$ as $p^2 \rightarrow 0$). Corresponding to (2.90a) and (2.90b), according to (2.86), are scaling operators belonging to \mathcal{M} satisfying (2.82)

$$1 \cdot \Phi_\delta = 1 \cdot \varphi + \frac{1}{2 - \eta} 1 \cdot G \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_*[\varphi], \quad \lambda = -\frac{1}{2}(d + 2 - \eta), \quad (2.91a)$$

$$1 \cdot \Phi_{d-\delta} = 1 \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_*[\varphi], \quad \lambda = -\frac{1}{2}(d - 2 + \eta). \quad (2.91b)$$

For $\eta = 0$, $h \rightarrow \mathcal{G}_0^{-1}$, and (2.90a) becomes

$$\Phi_{\delta_0} = \varphi - (\mathcal{G}_0 - \mathcal{G}) \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_*. \quad (2.92)$$

Imposing

$$\mathcal{G}_0^{-1} \cdot \Phi_{\delta_0} = Z \Phi_{d-\delta_0}, \quad (2.93)$$

restricts \mathcal{S}_* to a Gaussian form

$$\mathcal{S}_*[\varphi] = \frac{1}{2} \varphi \cdot F_* \cdot \varphi + c_* V, \quad F_* = (1 - Z) (\mathcal{G}_0 - (1 - Z) \mathcal{G})^{-1}, \quad (2.94)$$

where $c_* V = \text{tr}(G \cdot F_*)/2d$ is determined by requiring $E[\varphi] = 0$, according to (2.73). However when $d - \delta_0 = n\delta_0$ (2.93) may be relaxed.

The space of operators $\{\mathcal{O}\} \subset \mathcal{M}$ includes a subspace $\{\mathcal{O}_\psi\}$ of redundant operators of the form

$$\mathcal{O}_\psi = \psi \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* + \psi \cdot \mathcal{G}^{-1} \cdot \varphi - \frac{\delta}{\delta \varphi} \cdot \psi = \psi \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* - \frac{\delta}{\delta \varphi} \cdot \psi, \quad (2.95)$$

for $\psi[\phi; x] \in V_\phi$ and \mathcal{S}_* related to S_* as in (2.20). Redundant operators as in (2.95) may be regarded as associated with the freedom of choice for Ψ_t in the RG flow equations as in (2.11). Clearly $1 \cdot \Phi_{d-\delta}$ in (2.91b) is a redundant operator.

To show that $\{\mathcal{O}_\psi\}$ form a closed subspace we consider

$$\begin{aligned} \Delta_{\mathcal{S}_*} \left(\psi \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_* - \frac{\delta}{\delta\varphi} \cdot \psi \right) \\ = (\Delta_{\mathcal{S}_*, \text{loc}} \psi - D^{(\delta)} \psi) \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_* - \frac{\delta}{\delta\varphi} \cdot (\Delta_{\mathcal{S}_*, \text{loc}} \psi - D^{(\delta)} \psi) - \eta \psi \cdot \mathcal{G}^{-1} \cdot \varphi, \end{aligned} \quad (2.96)$$

with $\Delta_{\mathcal{S}_*, \text{loc}}$ as in (2.85). Furthermore, using (2.23),

$$\begin{aligned} \Delta_{\mathcal{S}_*} (\psi \cdot \mathcal{G}^{-1} \cdot \varphi) &= (\Delta_{\mathcal{S}_*, \text{loc}} \psi - D^{(\delta)} \psi - G \cdot \mathcal{G}^{-1} \cdot \psi) \cdot \mathcal{G}^{-1} \cdot \varphi \\ &\quad - (G \cdot \mathcal{G}^{-1} \cdot \psi) \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_* + \frac{\delta}{\delta\varphi} \cdot (G \cdot \mathcal{G}^{-1} \cdot \psi) + \eta \psi \cdot \mathcal{G}^{-1} \cdot \varphi. \end{aligned} \quad (2.97)$$

With the definition (2.95) combining (2.96) and (2.97) then gives

$$\Delta_{\mathcal{S}_*} \mathcal{O}_\psi = \mathcal{O}_{\tilde{\Delta}_{\mathcal{S}_*} \psi}, \quad (2.98)$$

for

$$\tilde{\Delta}_{\mathcal{S}_*} \psi = \Delta_{\mathcal{S}_*, \text{loc}} \psi - G \cdot \mathcal{G}^{-1} \cdot \psi - D^{(\delta)} \psi. \quad (2.99)$$

It is then clear from (2.98) that for redundant operators (2.82) is equivalent to

$$\tilde{\Delta}_{\mathcal{S}_*} \psi = \lambda \psi. \quad (2.100)$$

Using (2.27) and (2.76) then from (2.99)

$$\mathcal{G}_0 \cdot \mathcal{G}^{-1} \cdot \tilde{\Delta}_{\mathcal{S}_*} \psi = \Delta_{\mathcal{S}_*, \text{loc}} \mathcal{G}_0 \cdot \mathcal{G}^{-1} \cdot \psi - D^{(\delta)} (\mathcal{G}_0 \cdot \mathcal{G}^{-1} \cdot \psi). \quad (2.101)$$

Hence, for any local operator Φ_Δ satisfying (2.84), (2.101) ensures that

$$\Delta_{\mathcal{S}_*} \mathcal{O}_{\mathcal{G} \cdot \mathcal{G}_0^{-1} \cdot \Phi_\Delta} = (\Delta - \delta) \mathcal{O}_{\mathcal{G} \cdot \mathcal{G}_0^{-1} \cdot \Phi_\Delta}, \quad (2.102)$$

giving for each Φ_Δ an associated redundant operator.

For non redundant operators \mathcal{O} , which are defined so as to be linearly independent of all $\{\mathcal{O}_\psi\}$, (2.82) may be relaxed to require only

$$\Delta_{\mathcal{S}_*} \mathcal{O} = \lambda \mathcal{O} + \mathcal{O}_\chi, \quad (2.103)$$

for some redundant operator \mathcal{O}_χ of the form (2.95), as (2.103) may be reduced to (2.82) since

$$\Delta_{\mathcal{S}_*} (\mathcal{O} + \mathcal{O}_\psi) = \lambda (\mathcal{O} + \mathcal{O}_\psi) \quad \text{for} \quad (\tilde{\Delta}_{\mathcal{S}_*} - \lambda) \psi = \chi. \quad (2.104)$$

Assuming no accidental degeneracies of the eigenvalues of non redundant and redundant operators $\tilde{\Delta}_{\mathcal{S}_*} - \lambda$ is then invertible, so long as \mathcal{O} is not redundant, and ψ may be found in terms of χ .

Using the operator $\Delta_{\mathcal{S}_*}$ equations for the variation in the fixed point action \mathcal{S}_* consequent on variations in the initial Ψ which determine the RG equation may be found. Here Ψ is given by (2.17) and (2.22). For a variation of η in (2.73)

$$\Delta_{\mathcal{S}_*} \delta \mathcal{S}_* = -\frac{1}{2} \delta \eta \mathcal{O}_\varphi, \quad (2.105)$$

and for variations in G and hence \mathcal{G} , which are related by (2.23), then

$$\Delta_{\mathcal{S}_*} \left(\delta \mathcal{S}_* + \frac{1}{2} \varphi \cdot \delta \mathcal{G}^{-1} \cdot \varphi - \delta \text{tr}(G \cdot \mathcal{G}^{-1})/2d \right) = \Delta_{\mathcal{S}_*} \delta \mathcal{S}_* = \mathcal{O}_\psi, \quad (2.106)$$

for

$$\psi = \frac{1}{2} \delta G \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_* + \frac{1}{2} \delta G \cdot \mathcal{G}^{-1} \cdot \varphi - G \cdot \delta \mathcal{G}^{-1} \cdot \varphi, \quad (2.107)$$

and $\Delta_{\mathcal{S}_*}$ given by (2.83). Solutions of (2.105) and (2.106) lie within the space of redundant operators. However as discussed subsequently $\Delta_{\mathcal{S}_*}$ is expected to have in general a non trivial cokernel at non trivial fixed points. The quantisation of η at the fixed point requires that (2.105) should have no solution for $\delta \mathcal{S}_*$.

Corresponding to the transformation $\mathcal{S}_* \rightarrow T_*$ in (2.78), which gives rise to a linear fixed point equation as in (2.80), there is an associated transformation for any $\Phi \in \mathcal{M}'$ with $\Phi(x) \rightarrow P_\Phi(x)$ given by

$$P_\Phi(x) e^{T_*[\varphi]} = e^{\frac{1}{2} \varphi \cdot h \cdot \varphi} e^{\mathcal{Y}} \left(e^{-\mathcal{S}_*[\varphi]} \Phi(x) \right). \quad (2.108)$$

This transformation may also be extended to $\mathcal{O} \rightarrow P_\mathcal{O}$ for $\mathcal{O} \in \mathcal{M}$ but as shown later this need not always be well defined. Directly from the definition (2.108) the identity is invariant, $P_1 = 1$. Using

$$\begin{aligned} e^{-\mathcal{S}_*[\varphi]} \Delta_{\mathcal{S}_*, \text{loc}} \Phi &= \left(D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi} \right) (e^{-\mathcal{S}_*[\varphi]} \Phi) \\ &\quad + E[\varphi] e^{-\mathcal{S}_*[\varphi]} \Phi, \end{aligned} \quad (2.109)$$

for $E[\varphi]$ given by (2.73), with (2.44) and (2.56) then

$$e^{\frac{1}{2} \varphi \cdot h \cdot \varphi} e^{\mathcal{Y}} \left(e^{-\mathcal{S}_*[\varphi]} \Delta_{\mathcal{S}_*, \text{loc}} \Phi(x) \right) = \left(D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) e^{\frac{1}{2} \varphi \cdot h \cdot \varphi} e^{\mathcal{Y}} \left(e^{-\mathcal{S}_*[\varphi]} \Phi(x) \right). \quad (2.110)$$

Hence (2.108) gives

$$D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} P_\Phi = P_{\Delta_{\mathcal{S}_*, \text{loc}} \Phi} \quad (2.111)$$

For an eigenoperator satisfying (2.84) then

$$D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} P_{\Phi_\Delta} = D^{(\Delta)} P_{\Phi_\Delta}. \quad (2.112)$$

This equation is independent of \mathcal{S}_* and does not therefore determine Δ in any non trivial fashion, which is consistent with the transformation $\Phi \rightarrow P_\Phi$ not being invertible as $e^{-\mathcal{Y}}$ acting on arbitrary functionals is ill defined.

To consider the transformation of the exact eigenoperators Φ_δ and $\Phi_{d-\delta}$, defined in (2.90a) and (2.90b), we first obtain from (2.108) and (2.78)

$$\begin{aligned} P_\varphi &= (1 - \mathcal{G} \cdot h) \cdot \varphi + \mathcal{G} \cdot \frac{\delta}{\delta\varphi} T_*[\varphi], \\ P_{\frac{\delta}{\delta\varphi} \mathcal{S}_*[\varphi]} &= h \cdot \varphi - \frac{\delta}{\delta\varphi} T_*[\varphi]. \end{aligned} \quad (2.113)$$

Hence

$$P_{\Phi_\delta} = \mathcal{G}_0 \cdot \frac{\delta}{\delta\varphi} T_*[\varphi], \quad (2.114a)$$

$$P_{\Phi_{d-\delta}} = \mathcal{G}_0^{-1} \cdot \varphi. \quad (2.114b)$$

Using (2.80) and (2.76) it is straightforward to check that

$$D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta\varphi} P_{\Phi_\delta} = D^{(\delta)} P_{\Phi_\delta}, \quad D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta\varphi} P_{\Phi_{d-\delta}} = D^{(d-\delta)} P_{\Phi_{d-\delta}}. \quad (2.115)$$

(2.114a), (2.114b) demonstrate that the transformation $\mathcal{O} \rightarrow P_{\mathcal{O}}$, given by (2.108), does not extend to arbitrary $\mathcal{O} \in \mathcal{M}$, since from (2.114a) $1 \cdot P_{\Phi_\delta}$ is singular while from (2.114b) $1 \cdot P_{\Phi_{d-\delta}}$ is zero while there are no such problems for $1 \cdot \Phi_\delta$ and $1 \cdot \Phi_{d-\delta}$. However for a redundant operator, as in (2.95), (2.108) gives

$$P_{\mathcal{O}_\psi} = \varphi \cdot \mathcal{G}^{-1} \cdot P_\psi. \quad (2.116)$$

Using (2.23)

$$\begin{aligned} D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta\varphi} (\varphi \cdot \mathcal{G}^{-1} \cdot P_\psi) &= \varphi \cdot \mathcal{G}^{-1} \cdot \left(D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta\varphi} - D^{(\delta)} - \mathcal{G} \cdot \mathcal{G}^{-1} \cdot \right) P_\psi \\ &= \varphi \cdot \mathcal{G}^{-1} \cdot P_{\tilde{\Delta}_{\mathcal{S}_*} \psi}, \end{aligned} \quad (2.117)$$

as expected according to (2.98) with $\tilde{\Delta}_{\mathcal{S}_*}$ defined in (2.99).

2.5 Zero Modes

Amongst the scaling operators appearing in the asymptotic expansion (2.81) the zero modes, or marginal operators, \mathcal{Z} for which $\lambda_{\mathcal{Z}} = 0$ so that

$$\Delta_{\mathcal{S}_*} \mathcal{Z} = 0, \quad (2.118)$$

are of especial interest. The fixed point action $\mathcal{S}_*[\varphi]$ is then arbitrary to the extent $\mathcal{S}_*[\varphi] \sim \mathcal{S}_*[\varphi] - \varepsilon \mathcal{Z}[\varphi]$ for infinitesimal ε . If this is integrable there is an associated line of fixed points. For simple scalar theories a marginal operator \mathcal{Z} can be obtained by considering particular reparameterisations of the scalar field φ but are redundant since they are then removable by such a redefinition, or equivalently are zero subject to the dynamical field equations. If the line of fixed points generated by this redundant \mathcal{Z} is parameterised by a variable a then all $\mathcal{S}_*[\varphi; a]$ are equivalent.

For a redundant operator then

$$\mathcal{Z} = \mathcal{O}_{\psi_{\mathcal{Z}}} , \quad (2.119)$$

for appropriate $\psi_{\mathcal{Z}}$. Using (2.102) and (2.90a) the zero mode operator is determined in terms of Φ_{δ} by taking

$$\begin{aligned} \psi_{\mathcal{Z}} &= \mathcal{G} \cdot \mathcal{G}_0^{-1} \cdot \Phi_{\delta} = \mathcal{G} \cdot h \cdot \varphi + (\mathcal{G} \cdot h \cdot \mathcal{G} - \mathcal{G}) \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_*[\varphi] \\ &= \varphi + (\mathcal{G} \cdot h \cdot \mathcal{G} - \mathcal{G}) \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_*[\varphi] . \end{aligned} \quad (2.120)$$

This result for $\psi_{\mathcal{Z}}$ gives, using (2.95), an expression for $\mathcal{Z}[\varphi]$ identical with the operator constructed by O'Dwyer and Osborn [18].

Although $\psi_{\mathcal{Z}}$ and hence \mathcal{Z} is relatively complicated under the transformation $\mathcal{Z} \rightarrow P_{\mathcal{Z}}$ given by (2.108) there are significant simplifications. Using (2.120) to express $\psi_{\mathcal{Z}}$ in terms of P_{δ} and then (2.114a) with (2.116) gives

$$P_{\mathcal{Z}}[\varphi] = \varphi \cdot \frac{\delta}{\delta\varphi} T_*[\varphi] . \quad (2.121)$$

This demonstrates that letting $\mathcal{S}_*[\varphi] \rightarrow \mathcal{S}_*[\varphi] - \varepsilon \mathcal{Z}[\varphi]$ induces the associated transformation $T_*[\varphi] \rightarrow T_*[\varphi] + \varepsilon \varphi \cdot \frac{\delta}{\delta\varphi} T_*[\varphi]$ and hence that the line IR fixed points generated by \mathcal{Z} corresponds to a simple rescaling of φ in T_* , $T_*[\varphi] \sim T_*[\lambda\varphi]$ for all $\lambda > 0$.

The existence of a zero mode \mathcal{Z} satisfying (2.118) is potentially crucial in ensuring that the anomalous dimension η is determined at an IR fixed point. This requires that (2.105), corresponding to varying η , has no solution. The presence of a zero mode demonstrated that $\Delta_{\mathcal{S}_*}$ has a non trivial kernel while the lack of solutions to (2.105) in general would be a consequence of a non zero cokernel. These would be identical if it were possible to construct a scalar product with respect to which $\Delta_{\mathcal{S}_*}$ was self adjoint.

Associated with the marginal operator \mathcal{Z} there is a corresponding local operator $\Phi_{\mathcal{Z}}(x)$ satisfying (2.84) with $\Delta = d$. To determine this explicitly we first construct a bilocal functional $\mathcal{F}[\varphi; x, y]$ satisfying

$$\Delta_{\mathcal{S}_*, \text{loc}} \mathcal{F}(x, y) = D_x^{(d-\delta)} \mathcal{F}(x, y) + \mathcal{F}(x, y) \overleftarrow{D}_y^{(\delta)} . \quad (2.122)$$

Suppressing the x, y arguments this is satisfied by taking

$$\begin{aligned} \mathcal{F} &= \Phi_{d-\delta} \Phi_{\delta} + \mathcal{G}_0^{-1} \cdot \mathcal{G} \cdot S_*^{(2)} \cdot (1 - \mathcal{G} \cdot h) \cdot \mathcal{G}_0 + c \mathcal{I} \\ &= \left(\Phi_{d-\delta} - \mathcal{G}_0^{-1} \cdot \mathcal{G} \cdot \frac{\delta}{\delta\varphi} \right) \Phi_{\delta} + (c + 1) \mathcal{I} , \end{aligned} \quad (2.123)$$

where $\Phi_{d-\delta}, \Phi_{\delta}$ are given by (2.90b), (2.90a) albeit the latter in (2.123) is rewritten in terms of its transpose

$$\Phi_{\delta} = \varphi \cdot \mathcal{G}^{-1} \cdot \mathcal{G}_0 - \frac{\delta}{\delta\varphi} S_* \cdot (1 - \mathcal{G} \cdot h) \cdot \mathcal{G}_0 , \quad (2.124)$$

and $S_*^{(2)}$ is also the symmetric bilocal functional defined by

$$S_*^{(2)}[\varphi; x, y] = \frac{\delta^2}{\delta\varphi(x) \delta\varphi(y)} S_*[\varphi] . \quad (2.125)$$

In (2.123) $\mathcal{I} \rightarrow \delta^d(x-y)$ which satisfies (2.122) trivially since $\Delta_{\mathcal{S}^*,\text{loc}}\mathcal{I} = 0$, so that c is, for the moment, an unconstrained constant. To verify that (2.123) satisfies (2.122) it is sufficient to use

$$\Delta_{\mathcal{S}^*,\text{loc}} \Phi_{d-\delta} \Phi_\delta - D^{(d-\delta)} \Phi_{d-\delta} \Phi_\delta - \Phi_{d-\delta} \Phi_\delta \overleftarrow{D}^{(\delta)} = \Phi_{d-\delta} \frac{\overleftarrow{\delta}}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} \Phi_\delta, \quad (2.126)$$

and, from (2.73),

$$\begin{aligned} \Delta_{\mathcal{S}^*,\text{loc}} S_*^{(2)} - D^{(d-\delta)} S_*^{(2)} - S_*^{(2)} \overleftarrow{D}^{(d-\delta)} \\ = -\mathcal{G}^{-1} \cdot G \cdot S_*^{(2)} - S_*^{(2)} \cdot G \cdot \mathcal{G}^{-1} + S_*^{(2)} \cdot G \cdot S_*^{(2)}, \end{aligned} \quad (2.127)$$

together with identities such as (2.26), (2.62) and (2.76). In the same fashion as for $\Phi_{d-\delta}$ and Φ_δ , \mathcal{F} is also quasi-local. It is then sufficient from (2.122) to take

$$\Phi_{\mathcal{Z}}(x) = \mathcal{F}(x, x), \quad (2.128)$$

so long as the coincident limit is non singular. This determines c , if the leading term in $S_*^{(2)}$ is \mathcal{G}^{-1} then $c = -1$. Note that also

$$\mathcal{Z} = \text{tr}(\mathcal{F}). \quad (2.129)$$

If we consider $\Phi_{\mathcal{Z}} \rightarrow P_{\Phi_{\mathcal{Z}}}$, as determined by (2.108), then

$$P_{\Phi_{\mathcal{Z}}} = P_{\Phi_{d-\delta}} P_{\Phi_\delta}, \quad (2.130)$$

where $P_{\Phi_{d-\delta}} P_{\Phi_\delta}$ are given by (2.114a),(2.114b). In this case taking the coincident limit of $P_{\Phi_{d-\delta}}(x) P_{\Phi_\delta}(y)$ causes no problems and it is also trivial that $D^{(\delta-\eta)}\varphi \cdot \frac{\delta}{\delta\varphi} P_{\Phi_{\mathcal{Z}}} = D^{(d)} P_{\Phi_{\mathcal{Z}}}$ from (2.112) for $P_{\Phi_{d-\delta}}$ and P_{Φ_δ} . However it is non trivial that $P_{\Phi_{\mathcal{Z}}}$ corresponds to a quasi-local $\Phi_{\mathcal{Z}}$.

2.6 Solution of RG Equations with a Source

In quantum field theory it is natural to introduce a source term $e^{\hat{J} \cdot \phi}$, $\hat{J} \cdot \phi = \int d^d x \hat{J}(x) \phi(x)$, into the functional integral so that all correlation functions are obtained in terms of functional derivatives with respect to \hat{J} . In this context we therefore consider RG flow equations starting from $\hat{S}_{\Lambda_0}[\phi, \hat{J}] = \hat{S}_{\Lambda_0}[\phi] - \hat{J} \cdot \phi$, or following (2.9), (2.10) and (2.28),

$$\mathcal{S}_0[\varphi, J] = \mathcal{S}_0[\varphi] - J \cdot \varphi, \quad \hat{J}(x) = \Lambda^{\frac{1}{2}(d+2)} J(x\Lambda), \quad (2.131)$$

so that $\hat{J} \cdot \phi = J \cdot \varphi$. The RG flow equation (2.24) may be extended to the form

$$\begin{aligned} \left(\frac{\partial}{\partial t} + D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} + D^{(d-\delta)}J \cdot \frac{\delta}{\delta J} - dV \frac{\partial}{\partial V} \right) \mathcal{S}_t[\varphi, J] \\ = \frac{1}{2} \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi, J] \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi, J] - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi, J] - \frac{1}{2} \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1)). \end{aligned} \quad (2.132)$$

We here show how a solution for $\mathcal{S}_t[\varphi, J]$, with the initial condition (2.131), may be found in terms of $\mathcal{S}_t[\varphi]$ by taking

$$\mathcal{S}_t[\varphi, J] = \mathcal{S}_t[\varphi + \mathcal{B}_t \cdot J] - \frac{1}{2} J \cdot \mathcal{C}_t \cdot J - J \cdot \mathcal{D}_t \cdot \varphi. \quad (2.133)$$

Substituting (2.133) into (2.132) and using (2.24) gives a solution so long as

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{B}_t - D^{(\delta)} \mathcal{B}_t - \mathcal{B}_t \overleftarrow{D}^{(\delta)} + G \cdot \mathcal{D}_t^T &= 0, \\ \frac{\partial}{\partial t} \mathcal{D}_t - D^{(\delta)} \mathcal{D}_t - \mathcal{D}_t \overleftarrow{D}^{(d-\delta)} + \eta \mathcal{B}_t^T \cdot \mathcal{G}^{-1} &= 0, \\ \frac{\partial}{\partial t} \mathcal{C}_t - \mathcal{D}^{(\delta)} \mathcal{C}_t - \mathcal{C}_t \overleftarrow{D}^{(\delta)} + \eta \mathcal{B}_t^T \cdot \mathcal{G}^{-1} \cdot \mathcal{B}_t + \mathcal{D}_t \cdot G \cdot \mathcal{D}_t^T &= 0, \end{aligned} \quad (2.134)$$

with similar definitions to (2.14), and to ensure (2.131)

$$\mathcal{B}_0 = 0, \quad \mathcal{C}_0 = 0, \quad \mathcal{D}_0 = 1. \quad (2.135)$$

It is easy to see that it is sufficient to take

$$\mathcal{C}_t = \mathcal{D}_t \cdot \mathcal{B}_t, \quad (2.136)$$

so that (2.134) may be reduced to just the coupled equations

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\mathcal{B}}_t(p) + (p \cdot \partial_p + 2 - \eta) \tilde{\mathcal{B}}_t(p) &= -2K'(p^2) \tilde{\mathcal{D}}_t(-p), \\ \frac{\partial}{\partial t} \tilde{\mathcal{D}}_t(p) + p \cdot \partial_p \tilde{\mathcal{D}}_t(p) &= -\eta \frac{p^2}{K(p^2)} \tilde{\mathcal{B}}_t(-p). \end{aligned} \quad (2.137)$$

From this we may obtain

$$\frac{d}{dt} \left(k(t) \tilde{\mathcal{D}}_t(e^t p) + e^{2t} \tilde{\mathcal{B}}_t(-e^t p) \right) = 0, \quad (2.138)$$

for

$$k(t) = \frac{K(e^{2t} p^2)}{p^2}. \quad (2.139)$$

Using the solution of (2.138) we find

$$\frac{d}{dt} \tilde{\mathcal{D}}_t(e^t p) - \eta \tilde{\mathcal{D}}_t(e^t p) = -\eta \frac{k(0)}{k(t)}. \quad (2.140)$$

It is then easy to obtain

$$\tilde{\mathcal{D}}_t(p) = \frac{K(e^{-2t} p^2)}{K(p^2)} (1 - \sigma_t(p^2)), \quad \tilde{\mathcal{B}}_t(p) = \frac{K(e^{-2t} p^2)}{p^2} \sigma_t(p^2), \quad (2.141)$$

for

$$\sigma_t(p^2) = -K(p^2) \int_0^t ds e^{\eta s} \frac{d}{ds} \frac{1}{K(e^{-2s} p^2)}. \quad (2.142)$$

When $\eta = 0$ this simplifies to

$$\tilde{\mathcal{D}}_t(p) = 1, \quad \tilde{\mathcal{B}}_t(p) = \frac{1}{p^2} \left(K(e^{-2t}p^2) - K(p^2) \right). \quad (2.143)$$

In the limit $t \rightarrow \infty$

$$\tilde{\mathcal{D}}_*(p) = \frac{1}{K(p^2)} (1 - \sigma_*(p^2)), \quad \tilde{\mathcal{B}}_*(p) = \frac{1}{p^2} \sigma_*(p^2), \quad (2.144)$$

where $\sigma_*(p^2)$ is defined in (2.64). Hence

$$\mathcal{S}_t[\varphi, J] \Big|_{t \rightarrow \infty} \rightarrow \mathcal{S}_*[\varphi, J] = \mathcal{S}_*[\varphi + \mathcal{B}_* \cdot J] - \frac{1}{2} J \cdot \mathcal{D}_* \cdot \mathcal{B}_* \cdot J - J \cdot \mathcal{D}_* \cdot \varphi. \quad (2.145)$$

This existence of a fixed point action $\mathcal{S}_*[\varphi, J]$ for arbitrary J is a reflection that J only couples to φ and the result essentially corresponds just to a shift in φ .

2.7 Relation of $T[\varphi]$ to Vacuum Functional $W[J]$

In standard quantum field theory it is conventional to introduce the generating functional W for all n -point connected correlation functions W is given by the contributions for all connected vacuum Feynman graphs in the presence of the source J so that

$$G^{(n)}(x_1, \dots, x_n) = \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} W[J] \Big|_{J=0}. \quad (2.146)$$

Standard functional manipulations [22] show that, with \mathcal{Y} given by (2.42),

$$e^{W[J]} = e^{\mathcal{Y}} e^{-\mathcal{S}_0[\varphi] + J \cdot \varphi} \Big|_{\varphi=0}. \quad (2.147)$$

Following Rosten [10] we show how this is linked with $T[\varphi]$ defined by (2.47). It is straightforward to extend (2.58), starting from (2.132), to

$$\left(\frac{\partial}{\partial t} + D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} + D^{(d-\delta)} J \cdot \frac{\delta}{\delta J} - \frac{1}{2} \eta \varphi \cdot \mathcal{G}^{-1} \cdot \varphi \right) \left(e^{\mathcal{Y}} e^{-\mathcal{S}_t[\varphi, J]} \right) = 0, \quad (2.148)$$

which has a similar solution to (2.59)

$$e^{\mathcal{Y}} e^{-\mathcal{S}_t[\varphi, J]} = e^{T[\varphi_t, J_t] + \frac{1}{2} \varphi_t \cdot h \cdot \varphi_t - \frac{1}{2} \varphi \cdot h \cdot \varphi + c e^{dt} V}, \quad (2.149)$$

for φ_t as in (2.60) and

$$J_t(x) = e^{-\frac{1}{2}(d+2-\eta)t} J(e^{-t}x). \quad (2.150)$$

However using the Baker-Campbell-Hausdorff formula in the form

$$e^{\mathcal{Y}} e^{J \cdot \varphi} = e^{J \cdot \left(\varphi + \mathcal{G} \cdot \frac{\delta}{\delta \varphi} \right)} e^{\mathcal{Y}} = e^{\frac{1}{2} J \cdot \mathcal{G} \cdot J + J \cdot \varphi} e^{J \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}} e^{\mathcal{Y}}, \quad (2.151)$$

it follows from (2.131) and (2.59) that, for $t = 0$, $T[\varphi, J]$ is expressible just in terms of $T[\varphi]$

$$T[\varphi, J] = \frac{1}{2} J \cdot \mathcal{G} \cdot J + J \cdot \varphi + T[\varphi + \mathcal{G} \cdot J]. \quad (2.152)$$

so that in (2.149) for any $t \geq 0$

$$T[\varphi_t, J_t] = \frac{1}{2} J_t \cdot \mathcal{G} \cdot J_t + J_t \cdot \varphi_t + T[\varphi_t + \mathcal{G} \cdot J_t]. \quad (2.153)$$

The previous discussion of the behaviour at an IR fixed point can now be extended to $T[\varphi, J]$ since (2.77) becomes

$$T[\varphi_t, J_t] \underset{t \rightarrow \infty}{\sim} -e^{\eta t} \frac{1}{2} \varphi \cdot \mathcal{G}_0^{-1} \cdot \varphi + T_*[\varphi, J]. \quad (2.154)$$

Using (2.77) with (2.153) with

$$J_t \cdot \varphi_t = e^{\eta t} J \cdot \varphi, \quad J_t \cdot \mathcal{G} \cdot J_t \underset{t \rightarrow \infty}{\sim} e^{\eta t} J \cdot \mathcal{G}_0 \cdot J \quad (2.155)$$

since

$$\mathcal{G}(e^t y) \underset{t \rightarrow \infty}{\sim} e^{-(d-2)t} \mathcal{G}_0(y), \quad (2.156)$$

for \mathcal{G}_0 as in (2.75), then

$$T_*[\varphi, J] = T_*[\varphi + \mathcal{G}_0 \cdot J]. \quad (2.157)$$

In a similar fashion to (2.78)

$$e^{T_*[\varphi, J] - \frac{1}{2} \varphi \cdot h \cdot \varphi} = e^{\mathcal{Y}} e^{-\mathcal{S}_*[\varphi, J]}. \quad (2.158)$$

Compatibility of this result with (2.157) and the solution for $-\mathcal{S}_*[\varphi, J]$ in (2.145) may be checked by applying the Baker-Campbell-Hausdorff formula just as in (2.151) to give

$$\begin{aligned} e^{\mathcal{Y}} e^{-\mathcal{S}_*[\varphi, J]} &= e^{\frac{1}{2} J \cdot (\mathcal{D}_* \cdot \mathcal{B}_* + \mathcal{D}_* \cdot \mathcal{G} \cdot \mathcal{D}_*^T) \cdot J + J \cdot \mathcal{D}_* \cdot \varphi} e^{J \cdot \mathcal{D}_* \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}} e^{\mathcal{Y}} e^{-\mathcal{S}[\varphi + \mathcal{B}_* \cdot J]} \\ &= e^{\frac{1}{2} J \cdot (\mathcal{D}_* \cdot \mathcal{B}_* + \mathcal{D}_* \cdot \mathcal{G} \cdot \mathcal{D}_*^T) \cdot J + J \cdot \mathcal{D}_* \cdot \varphi} e^{J \cdot \mathcal{D}_* \cdot \mathcal{G} \cdot \frac{\delta}{\delta \varphi}} e^{T_*[\varphi + \mathcal{B}_* \cdot J] - \frac{1}{2} (\varphi + J \cdot \mathcal{B}_*^T) \cdot h \cdot (\varphi + \mathcal{B}_* \cdot J)}, \end{aligned} \quad (2.159)$$

as a consequence of $\mathcal{B}_* + \mathcal{G} \cdot \mathcal{D}_*^T = \mathcal{G}_0$, $\mathcal{D}_* = \mathcal{G}_0 \cdot h$ and $\mathcal{D}_* \cdot \mathcal{B}_* + \mathcal{D}_* \cdot \mathcal{G} \cdot \mathcal{D}_*^T = \mathcal{G}_0 \cdot h \cdot \mathcal{G}_0$, noting that, from (2.64), $\tilde{h}(p) = (1 - \sigma_*(p^2))/\tilde{\mathcal{G}}(p)$.

For the generating functional for connected correlation functions $W[J]$, (2.147) and (2.149), (2.152) give

$$W[J] = \frac{1}{2} J \cdot \mathcal{G} \cdot J + T[\mathcal{G} \cdot J], \quad (2.160)$$

and using (2.77)

$$W[J_t] \underset{t \rightarrow \infty}{\sim} W_*[J] = T_*[\mathcal{G}_0 \cdot J], \quad (2.161)$$

This implies for the correlation functions defined in (2.146)

$$G^{(n)}(e^t x_1, \dots, e^t x_n) \underset{t \rightarrow \infty}{\sim} e^{-\frac{1}{2} n(d-2+\eta)t} G_*^{(n)}(x_1, \dots, x_n), \quad (2.162)$$

with, from (2.161),

$$G_*^{(n)}(x_1, \dots, x_n) = \prod_{r=1}^n \int d^d x'_r \mathcal{G}_0(y_r) \frac{\delta^n}{\delta \varphi(x'_1) \dots \delta \varphi(x'_n)} T_*[\varphi] \Big|_{\varphi=0}, \quad (2.163)$$

for $y_r = x_r - x'_r$.

Since (2.162) requires $G_*^{(n)}(e^t x_1, \dots, e^t x_n) = e^{-\frac{1}{2}n(d-2+\eta)t} G_*^{(n)}(x_1, \dots, x_n)$ then this is just the expected scaling behaviour at an IR critical point if η is identified with the ϕ anomalous dimension.

2.8 Gaussian Solution

Although perhaps somewhat trivial it is illustrative to consider a Gaussian solution which is quadratic in the fields of the form

$$\mathcal{S}_t[\varphi] = \frac{1}{2} \varphi \cdot F_t \cdot \varphi + c_t V = \frac{1}{2(2\pi)^d} \int d^d p \tilde{\varphi}(p) \tilde{F}_t(p) \tilde{\varphi}(-p) + c_t V, \quad (2.164)$$

where $F_t = F_t^T$. Substituting (2.164) into (2.24), or equivalently (2.32), gives an evolution equations for F_t and c_t ,

$$\frac{\partial}{\partial t} \tilde{F}_t(p) = (2 - \eta - p \cdot \partial_p) \tilde{F}_t(p) + 2K'(p^2) \tilde{F}_t(p)^2 - \eta \frac{p^2}{K(p^2)}, \quad (2.165a)$$

$$\left(\frac{\partial}{\partial t} - d \right) c_t V = -\frac{1}{2} \text{tr}(G \cdot F_t - \eta 1). \quad (2.165b)$$

(2.165a) can be rewritten as

$$\frac{d}{dt} \left(e^{-(2-\eta)t} \tilde{F}_t(e^t p) + \frac{e^{\eta t}}{k(t)} \right) = e^{-\eta t} \dot{k}(t) \left(\left(e^{-(2-\eta)t} \tilde{F}_t(e^t p) \right)^2 - \frac{e^{2\eta t}}{k(t)^2} \right), \quad (2.166)$$

with $k(t)$ by (2.139). This may be solved for any arbitrary initial $\tilde{F}_0(p)$ giving

$$\frac{p^2/K(p^2)}{\tilde{F}_t(p) + p^2/K(p^2)} - \sigma_t(p^2) = e^{\eta t} \frac{K(p^2)}{K(e^{-2t}p^2)} \frac{p^2/K(e^{-2t}p^2)}{e^{2t}\tilde{F}_0(e^{-t}p) + p^2/K(e^{-2t}p^2)}, \quad (2.167)$$

for σ_t given by (2.142). The corresponding solution of (2.165b) is trivial

$$c_t V = e^{dt} \left(c_0 V - \frac{1}{2} \int_0^t dt' e^{-dt'} \text{tr}(G \cdot F_{t'} - \eta 1) \right). \quad (2.168)$$

For $\eta = 0$ (2.167) may be simplified to

$$\frac{1}{\tilde{F}_t(p)} = \frac{e^{-2t}}{\tilde{F}_0(e^{-t}p)} + \frac{1}{p^2} \left(K(e^{-2t}p^2) - K(p^2) \right). \quad (2.169)$$

For locality it is crucial to assume $\tilde{F}_0(p)$ is analytic in p for $p \approx 0$.

If $t \rightarrow \infty$ then $\mathcal{S}_t \rightarrow \mathcal{S}_*$ where

$$\mathcal{S}_*[\varphi] = \frac{1}{2} \varphi \cdot F_* \cdot \varphi + c_* V, \quad (2.170)$$

and, for the limit $c_t \rightarrow c_*$ to exist, it is necessary that c_0 is fine tuned to cancel any e^{dt} terms giving

$$c_* V = \frac{1}{2d} \text{tr}(G \cdot F_* - \eta 1). \quad (2.171)$$

For $\tilde{F}_0(0) \neq 0$, and assuming $\eta < 2$, from (2.167)

$$\tilde{F}_*(p) = \lim_{t \rightarrow \infty} \tilde{F}_t(p) = \frac{p^2}{K(p^2)} \left(\frac{1}{\sigma_*(p^2)} - 1 \right), \quad (2.172)$$

which is independent of the initial $\tilde{F}_0(p)$. Since, from (2.65), $\sigma_*(p^2) \propto p^2$, as $p^2 \rightarrow 0$, $\tilde{F}_*(0) > 0$ so this fixed point does not lead to any IR long range order. It corresponds to the trivial high temperature fixed point described in the introduction. If $\tilde{F}_0(0) = 0$, which defines the critical surface for Gaussian theories, so that

$$\tilde{F}_0(p) = \frac{1}{z} p^2 + O((p^2)^2), \quad (2.173)$$

and $\eta = 0$ also then (2.169) gives the limit, depending only on z ,

$$\tilde{F}_*(p) = \frac{p^2}{1 + z - K(p^2)}. \quad (2.174)$$

This gives rise to an IR fixed point with long range order, the need for $\eta = 0$ was also made clear by Comellas [20]. For $\tilde{F}_t(p)$ to be non singular it is necessary that $z > 0$. The solutions for F_* in (2.170) are then

$$F_* = \begin{cases} (1 - h \cdot \mathcal{G})^{-1} \cdot h, & \eta \neq 0, \\ ((1 + z)\mathcal{G}_0 - \mathcal{G})^{-1}, & \eta = 0. \end{cases} \quad (2.175)$$

Hence from (2.90a) and (2.92)

$$\Phi_\delta = \begin{cases} 0, & \eta \neq 0, \\ z \mathcal{G}_0 \cdot ((1 + z)\mathcal{G}_0 - \mathcal{G})^{-1} \cdot \varphi, & \eta = 0, \delta = \delta_0. \end{cases} \quad (2.176)$$

For $\eta \neq 0$ therefore $\mathcal{Z} = 0$, corresponding to the lack of any condition determining η . For $\eta = 0$ using (2.175) in (2.170) is identical with (2.94) for $Z = z/(1 + z)$.

The result (2.176) can be used to verify the previous formula for zero modes since from (2.120)

$$\psi_{\mathcal{Z}} = H \cdot \varphi, \quad H = z \mathcal{G} \cdot F_*. \quad (2.177)$$

and then (2.95) gives

$$\mathcal{Z}[\varphi] = \varphi \cdot H^T \cdot (F_* + \mathcal{G}^{-1}) \cdot \varphi - \text{tr}(H), \quad (2.178)$$

where

$$H^T \cdot (F_* + \mathcal{G}^{-1}) = z(1+z) F_* \cdot \mathcal{G}_0 \cdot F_* = -z(1+z) \frac{\partial}{\partial z} F_* . \quad (2.179)$$

Since $\tilde{H}(p) = zK(p^2)/(1+z-K(p^2))$ then inserting $1 = \partial_p \cdot p/d$ and integrating by parts gives

$$\text{tr}(H) = -z(1+z) \frac{2}{d} \frac{V}{(2\pi)^d} \int d^d p \frac{p^2 K'(p^2)}{(1+z-K(p^2))^2} = 2z(1+z) \frac{\partial}{\partial z} c_* V , \quad (2.180)$$

since, with $\eta = 0$,

$$c_* V = \frac{1}{2d} \text{tr}(G \cdot F_*) = \frac{1}{d} \frac{V}{(2\pi)^d} \int d^d p \frac{p^2 K'(p^2)}{1+z-K(p^2)} . \quad (2.181)$$

Hence

$$\mathcal{Z}[\phi] = -2z(1+z) \frac{\partial}{\partial z} \mathcal{S}_*[\varphi] , \quad (2.182)$$

in accord with z being a redundant parameter.

Clearly \mathcal{S}_* depends on the cut off function K , on the other hand T_* obtained from \mathcal{S}_* according to (2.78) is independent of K . To determine T_* and demonstrate this in the Gaussian case it is sufficient to use

$$e^{\mathcal{Y}} e^{-\frac{1}{2} \varphi \cdot F_* \cdot \varphi} = e^{-\frac{1}{2} \varphi \cdot (G + F_*^{-1})^{-1} \cdot \varphi - \frac{1}{2} \text{tr} \ln(1 + G \cdot F_*)} . \quad (2.183)$$

From (2.175)

$$(\mathcal{G} + F_*^{-1})^{-1} = \begin{cases} h , & \eta \neq 0 , \\ \frac{1}{1+z} \mathcal{G}_0^{-1} , & \eta = 0 , \end{cases} \quad (2.184)$$

and assuming

$$\frac{1}{2} \text{tr} \ln(1 + \mathcal{G} \cdot F_*) + c_* V = 0 , \quad (2.185)$$

then, since for $\eta = 0$, $h = \mathcal{G}_0^{-1}$, (2.78) gives

$$T_*[\varphi] = \begin{cases} 0 , & \eta \neq 0 , \\ \frac{z}{1+z} \frac{1}{2} \varphi \cdot \mathcal{G}_0^{-1} \cdot \varphi , & \eta = 0 . \end{cases} \quad (2.186)$$

For $\eta = 0$ this result shows that

$$2z(1+z) \frac{\partial}{\partial z} T_*[\varphi] = \varphi \cdot \frac{\partial}{\partial \varphi} T_*[\varphi] = P_Z[\varphi] , \quad (2.187)$$

in accord with the expression (2.121) for the zero mode. To verify (2.185), with $c_* V$ given by (2.171), we note that

$$\frac{1}{2} \text{tr} \ln(1 + \mathcal{G} \cdot F_*) = \begin{cases} -\frac{V}{2(2\pi)^d} \int d^d p \ln(1 - \tilde{h}(p)K(p^2)/p^2) , & \eta \neq 0 , \\ \frac{V}{2(2\pi)^d} \int d^d p \ln(1 + K(p^2)/(1+z-K(p^2))) , & \eta = 0 , \end{cases} \quad (2.188)$$

and then insert $1 = \partial_p \cdot p/d$ and integrate by parts. For $\eta \neq 0$ it is necessary to use $p \cdot \partial_p (\tilde{h}(p)K(p^2)/p^2) = -\eta(1 - \tilde{h}(p)K(p^2)/p^2) + \tilde{h}(p)2K'(p^2)$.

Of course for the Gaussian fixed point all critical exponents can be obtained. To obtain the scaling dimensions Δ for local scalar operators $\Phi[\varphi; x]$ we consider the transformation (2.108) for \mathcal{S}_* given by (2.170). Extending (2.183), and using the results already obtained for T_* , ensures that (2.108) becomes in this case

$$P_\Phi[\varphi] = e^{\varphi \cdot \ln \mathcal{G}^{-1} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} e^{\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} \Phi[\varphi] = e^{\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G} \cdot \mathcal{G}_*^{-1} \cdot \mathcal{G} \cdot \frac{\delta}{\delta\varphi}} \Phi[\mathcal{G}_* \cdot \mathcal{G}^{-1} \cdot \varphi], \quad (2.189)$$

for

$$\mathcal{G}_* = \mathcal{G} - \mathcal{G} \cdot (\mathcal{G} + F_*^{-1})^{-1} \cdot \mathcal{G}, \quad \mathcal{G}_*^{-1} = F_* + \mathcal{G}^{-1}. \quad (2.190)$$

For $\eta = 0$

$$\mathcal{G}_* = \mathcal{G} - \frac{1}{1+z} \mathcal{G} \cdot \mathcal{G}_0^{-1} \cdot \mathcal{G}. \quad (2.191)$$

In this case (2.112) becomes

$$D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta\varphi} P_{\Phi_\Delta}[\varphi] = D^{(\Delta)} P_{\Phi_\Delta}[\varphi]. \quad (2.192)$$

With the basis of local operators for $\mathcal{M}'_{\text{local}}$ provided by (2.40)

$$D^{(\delta_0)} \varphi \cdot \frac{\delta}{\delta\varphi} P_{n,s}[\varphi; x] = D^{(\Delta_{n,s})} P_{n,s}[\varphi; x], \quad \Delta_{n,s} = n \delta_0 + s, \quad n, s = 0, 1, 2, \dots \quad (2.193)$$

Identifying $P_{\Phi_{n,s}} = P_{n,s}$ the corresponding operators $\Phi_{n,s}$ are obtained by inverting (2.189)

$$\begin{aligned} \Phi_{n,s}[\varphi] &= e^{-\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} e^{-\varphi \cdot \ln \mathcal{G}^{-1} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} P_{n,s}[\varphi] = e^{-\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} P_{n,s}[\mathcal{G} \cdot \mathcal{G}_*^{-1} \cdot \varphi] \\ &= \mathcal{N}_{\mathcal{G}_*} [P_{n,s}[\mathcal{G} \cdot \mathcal{G}_*^{-1} \cdot \varphi]]. \end{aligned} \quad (2.194)$$

The inversion is well defined acting on monomials of finite order in φ . As special cases

$$\begin{aligned} P_{1,0}[\varphi] = \varphi &\Rightarrow \Phi_{1,0} = \mathcal{G} \cdot \mathcal{G}_*^{-1} \cdot \varphi = \frac{1+z}{z} \Phi_{\delta_0}, \\ P_{1,2}[\varphi] = \mathcal{G}_0^{-1} \cdot \varphi &\Rightarrow \Phi_{1,2} = \mathcal{G}_0^{-1} \cdot \mathcal{G} \cdot \mathcal{G}_*^{-1} \cdot \varphi = \Phi_{d-\delta_0}, \end{aligned} \quad (2.195)$$

where Φ_{δ_0} , $\Phi_{d-\delta_0}$ are given by (2.92), (2.90b) with (2.170) and (2.175). In this case we may take $\mathcal{Z} = 1 \cdot \Phi_{2,2}$.

A basis of redundant operators may also be obtained from (2.116) by taking

$$\begin{aligned} \mathcal{O}_{\psi_{n,s}}[\varphi] &= e^{-\frac{1}{2} \frac{\delta}{\delta\varphi} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} e^{-\varphi \cdot \ln \mathcal{G}^{-1} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta\varphi}} (\varphi \cdot \mathcal{G}_0^{-1} \cdot P_{n,s}[\varphi]) \\ \text{for } \psi_{n,s} &= \mathcal{G} \cdot \mathcal{G}_0^{-1} \cdot P_{n,s}. \end{aligned} \quad (2.196)$$

In this basis $\mathcal{Z} = \mathcal{O}_{\psi_{1,0}}$ for $P_{1,0}$ as in (2.195).

For $\eta \neq 0$ (2.190) with (2.184) give

$$\mathcal{G}(p)^{-1} \tilde{\mathcal{G}}_*(p) = \sigma_*(p^2) = \mathcal{O}(p^2) \quad \text{as } p^2 \rightarrow 0. \quad (2.197)$$

In (2.189) $e^{\varphi \cdot \ln \mathcal{G}^{-1} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta \varphi}}$ then generates contributions which are singular as $p \rightarrow 0$ and so are non local. To construct a local basis we now write

$$P_{\Phi_{n,s}} = e^{\varphi \cdot \ln \mathcal{G}_0^{-1} \cdot \frac{\delta}{\delta \varphi}} P_{n,s}. \quad (2.198)$$

This modification generated additional terms in the eigenvalue equation

$$D^{(\delta-\eta)} \varphi \cdot \frac{\delta}{\delta \varphi} P_{\Phi_{n,s}} = e^{\varphi \cdot \ln \mathcal{G}_0^{-1} \cdot \frac{\delta}{\delta \varphi}} D^{(\delta-\eta+2)} \varphi \cdot \frac{\delta}{\delta \varphi} P_{n,s} = D^{(\Delta_{n,s})} P_{\Phi_{n,s}}, \quad (2.199)$$

where now

$$\Delta_{n,s} = \frac{1}{2}(d+2-\eta)n + s. \quad (2.200)$$

Manifestly there is no zero mode in this case, and for $\eta < 2$ there are no relevant operators even in φ except for the identity.

Instead of (2.196) a basis of redundant operators is obtained for the trivial high temperature fixed point by taking

$$\mathcal{O}_{\psi_{n,s}}[\varphi] = e^{-\frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta \varphi}} e^{-\varphi \cdot \ln \mathcal{G}^{-1} \cdot \mathcal{G}_* \cdot \frac{\delta}{\delta \varphi}} (\varphi \cdot P_{\Phi_{n,s}}[\varphi]). \quad (2.201)$$

This shows that all scaling operators, with eigenvalues given by $\Delta_{n,s} - d$ with $\Delta_{n,s}$ as in (2.200), are redundant except when $n = s = 0$, when $\mathcal{O} \propto V$ and $\lambda = -d$, reflecting the triviality of this fixed point (similar results were described in [3]). The lack of any condition constraining η also relates to the absence of a zero mode.

2.9 Legendre Transform

An alternative simple form for an exact RG equation, valid outside any perturbation theory, was introduced by Wetterich [19] by considering the RG flow of the one particle generating functional Γ . This was shown to be equivalent to the standard Polchinski equation by Morris [14] where Γ is related to the action \mathcal{S} appearing in the Polchinski equation by a Legendre transform. Here we show how this extends to the case when the Polchinski equation is modified by allowing for the freedom to introduce the free parameter η which plays the role of an anomalous dimension as in (2.24). A related discussion was recently given by Rosten [21].

Before introducing a Legendre transformation the Polchinski equation (2.24) is first rewritten in terms of the functional trace (2.5) so that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - d V \frac{\partial}{\partial V} \right) \mathcal{S}_t[\varphi] \\ &= \frac{1}{2} \frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi] - \frac{1}{2} \text{tr}(G \cdot \mathcal{S}_t^{(2)}[\varphi]) - \frac{1}{2} \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1)), \end{aligned} \quad (2.202)$$

where now

$$\mathcal{S}_t^{(2)}[\varphi; x, y] = \frac{\delta^2}{\delta \varphi(x) \delta \varphi(y)} \mathcal{S}_t[\varphi], \quad \mathcal{S}_t^{(2)}[\varphi]^T = \mathcal{S}_t^{(2)}[\varphi]. \quad (2.203)$$

The Legendre transform determining Γ is then

$$\mathcal{S}_t[\varphi] = \Gamma_t[\Phi] + \frac{1}{2} \Phi \cdot \mathcal{R} \cdot \Phi + \frac{1}{2} \varphi \cdot \mathcal{Q} \cdot \varphi - \varphi \cdot \mathcal{I} \cdot \Phi, \quad \mathcal{R} = \mathcal{R}^T, \quad \mathcal{Q} = \mathcal{Q}^T, \quad (2.204)$$

where Φ is defined by

$$\frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi] - \mathcal{Q} \cdot \varphi = -\mathcal{I} \cdot \Phi. \quad (2.205)$$

The Legendre transform leads directly to

$$\frac{\delta}{\delta \Phi} \Gamma_t[\Phi] + \Phi \cdot \mathcal{R} = \varphi \cdot \mathcal{I}, \quad \frac{\partial}{\partial t} \Gamma_t[\Phi] = \frac{\partial}{\partial t} \mathcal{S}_t[\varphi]. \quad (2.206)$$

It is easy to further obtain

$$\mathcal{S}_t^{(2)}[\varphi] - \mathcal{Q} = -\mathcal{I} \cdot (\mathcal{R} + \Gamma_t^{(2)}[\Phi])^{-1} \cdot \mathcal{I}^T, \quad (2.207)$$

where

$$\Gamma_t^{(2)}[\Phi; x, y] = \frac{\delta^2}{\delta \Phi(x) \delta \Phi(y)} \Gamma_t[\Phi]. \quad (2.208)$$

If we omit a Φ -independent term $\frac{1}{2} \text{tr}(G \cdot \mathcal{Q} - \eta 1) \propto V$, which is reconsidered later but may be consistently neglected in obtaining an equation for Γ_t , the RG flow equation (2.202) then becomes

$$\left(\frac{\partial}{\partial t} + D^{(\delta)} \Phi \cdot \frac{\delta}{\delta \Phi} - dV \frac{\partial}{\partial V} \right) \Gamma_t[\Phi] = \frac{1}{2} \text{tr} \left(\mathcal{I}^T \cdot G \cdot \mathcal{I} \cdot (\mathcal{R} + \Gamma_t^{(2)}[\Phi])^{-1} \right) \quad (2.209)$$

so long as

$$\begin{aligned} & -D^{(\delta)} \left(\frac{\delta}{\delta \Phi} \Gamma_t[\Phi] \cdot \mathcal{I}^{-1} + \Phi \cdot \mathcal{R} \cdot \mathcal{I}^{-1} \right) \cdot (\mathcal{I} \cdot \Phi - \mathcal{H} \cdot \varphi) \\ & = \frac{1}{2} (\Phi \cdot \mathcal{I}^T - \varphi \cdot \mathcal{Q}) \cdot G \cdot (\mathcal{I} \cdot \Phi - \mathcal{Q} \cdot \varphi) - \frac{1}{2} \eta \varphi \cdot \mathcal{G}^{-1} \cdot \varphi + \frac{\delta}{\delta \Phi} \Gamma_t[\Phi] \cdot D\Phi, \end{aligned} \quad (2.210)$$

using (2.205) and (2.206). Eliminating φ through (2.206) leads to equations for \mathcal{Q}, \mathcal{I} and \mathcal{R} which can be reduced to

$$-D^{(d-\delta)} \mathcal{Q} - \mathcal{Q} \overleftarrow{D}^{(d-\delta)} = \mathcal{Q} \cdot G \cdot \mathcal{Q} - \eta \mathcal{G}^{-1}, \quad (2.211a)$$

$$-D^{(d-\delta)} \mathcal{I} - \mathcal{I} \overleftarrow{D}^{(d-\delta)} = \mathcal{Q} \cdot G \cdot \mathcal{I}, \quad (2.211b)$$

$$-D^{(d-\delta)} \mathcal{R} - \mathcal{R} \overleftarrow{D}^{(d-\delta)} = \mathcal{I}^T \cdot G \cdot \mathcal{I}, \quad (2.211c)$$

where \bar{D} is related to D as in (2.14). With (2.29) and (2.30) these become the differential equations

$$(p \cdot \partial_p - 2 + \eta) \tilde{\mathcal{Q}}(p) = 2K'(p^2) \tilde{\mathcal{Q}}(p)^2 - \eta \frac{p^2}{K(p^2)}, \quad (2.212a)$$

$$(p \cdot \partial_p - 2 + \eta) \tilde{\mathcal{I}}(p) = 2K'(p^2) \tilde{\mathcal{Q}}(p) \tilde{\mathcal{I}}(p), \quad (2.212b)$$

$$(p \cdot \partial_p - 2 + \eta) \tilde{\mathcal{R}}(p) = 2K'(p^2) \tilde{\mathcal{I}}(-p) \tilde{\mathcal{I}}(p). \quad (2.212c)$$

(2.212a) is similar to (2.165a) and can be solved in an analogous fashion

$$f(p^2) = (p^2)^{-1+\frac{1}{2}\eta} \tilde{\mathcal{Q}}(p) + \frac{(p^2)^{\frac{1}{2}\eta}}{K(p^2)} \Rightarrow f'(x) = x^{-\frac{1}{2}\eta} K'(x) f(x) \left(f(x) - \frac{2x^{\frac{1}{2}\eta}}{K(x)} \right), \quad (2.213)$$

or

$$\frac{d}{dx} \left(\frac{1}{f(x)K(x)^2} \right) = x^{-\frac{1}{2}\eta} \frac{d}{dx} \frac{1}{K(x)}. \quad (2.214)$$

Requiring analyticity for $p \approx 0$ determines a unique solution

$$\tilde{\mathcal{Q}}(p) = \frac{p^2}{K(p^2)} \left(\frac{1}{\sigma_*(p^2)} - 1 \right), \quad (2.215)$$

where σ_* is given by (2.64). (2.212b) is a linear homogeneous first order equation for $\tilde{\mathcal{I}}(p)$ which can be easily solved by integration. With an arbitrary choice for the overall scale we have

$$\tilde{\mathcal{I}}(p) = \frac{p^2}{\sigma_*(p^2)}. \quad (2.216)$$

With this solution (2.212c) becomes

$$\tilde{\mathcal{R}}(p) = (p^2)^{1-\frac{1}{2}\eta} r(p^2), \quad r'(x) = x^{\frac{1}{2}\eta} \frac{K'(x)}{\sigma_*(x)^2} = \frac{d}{dx} \left(x^{\frac{1}{2}\eta} \frac{K(x)}{\sigma_*(x)} \right), \quad (2.217)$$

so that, assuming analyticity for $p \approx 0$ again,

$$\tilde{\mathcal{R}}(p) = \frac{p^2 K(p^2)}{\sigma_*(p^2)}. \quad (2.218)$$

From the definition (2.64), for general $\eta < 2$, the asymptotic behaviour of $\sigma_*(p^2)$ is given by (2.65) and

$$\sigma_*(p^2) \rightarrow 1 \quad \text{as} \quad p^2 \rightarrow \infty, \quad (2.219)$$

assuming that $K(p^2) \rightarrow 0$ as $p^2 \rightarrow \infty$ faster than any inverse power and $K'(p^2) < 0$ for all p^2 . In consequence $\tilde{\mathcal{R}}(p)$, as defined by (2.218), is a finite positive constant for $p = 0$ and falls off rapidly for large p^2 . Reinstating the cut off scale Λ in (2.218) by dimensional considerations in the form

$$\tilde{\mathcal{R}}_\Lambda(p) = \frac{p^2 K(p^2/\Lambda^2)}{\sigma_*(p^2/\Lambda^2)}, \quad (2.220)$$

then we may define

$$\dot{\mathcal{R}} \equiv \frac{\partial}{\partial t} \mathcal{R}_\Lambda \Big|_{\Lambda=1} = -D^{(d-\delta)} \mathcal{R} - \mathcal{R} \overleftarrow{D}^{(d-\delta)} - \eta \mathcal{R} = \mathcal{I}^T \cdot G \cdot \mathcal{I} - \eta \mathcal{R}, \quad (2.221)$$

as a consequence of (2.211c) or (2.212c).

Using the above results the omitted term in (2.209) becomes

$$\frac{1}{2} \text{tr}(G \cdot \mathcal{Q} - \eta 1) = \frac{1}{2} \text{tr}(\dot{\mathcal{R}} \cdot \mathcal{R}^{-1} - G \cdot \mathcal{G}^{-1}), \quad (2.222)$$

which are removed by the natural redefinitions $\Gamma_t[\Phi] - \text{tr}(\dot{\mathcal{R}} \cdot \mathcal{R}^{-1})/2d \rightarrow \Gamma_t[\Phi]$ and also $\mathcal{S}_t[\varphi] - \text{tr}(G \cdot \mathcal{G}^{-1})/2d \rightarrow \mathcal{S}_t[\varphi]$. In terms of the full action S_t , related to \mathcal{S}_t as in (2.20), (2.204) now becomes

$$S_t[\varphi] = \Gamma_t[\Phi] + \frac{1}{2} (\Phi - \mathcal{G}_0 \cdot \mathcal{G}^{-1} \cdot \varphi) \cdot \mathcal{R} \cdot (\Phi - \mathcal{G}_0 \cdot \mathcal{G}^{-1} \cdot \varphi) - \frac{1}{2d} \text{tr}(\dot{\mathcal{R}} \cdot \mathcal{R}^{-1}), \quad (2.223)$$

where \mathcal{G}_0 is given in (2.75). Applying (2.221) in (2.209)

$$\left(\frac{\partial}{\partial t} + D^{(\delta)} \Phi \cdot \frac{\delta}{\delta \Phi} - dV \frac{\partial}{\partial V} \right) \Gamma_t[\Phi] = \frac{1}{2} \text{tr} \left((\dot{\mathcal{R}} + \eta \mathcal{R}) \cdot (\mathcal{R} + \Gamma_t^{(2)}[\Phi])^{-1} \right), \quad (2.224)$$

for

$$\tilde{\mathcal{R}}(p) = (p \cdot \partial_p - 2) \tilde{\mathcal{R}}(p). \quad (2.225)$$

(2.224) is just the standard form of the Wetterich RG equation [19] extended to include the parameter η . Assuming \mathcal{R} is an independent cut off function all dependence on η is explicit.

If there is a fixed point as $t \rightarrow \infty$ for a suitable choice of η then at the fixed point Γ_* must satisfy, by virtue of (2.209) and (2.221),

$$E[\Phi] \equiv \left(D^{(\delta)} \Phi \cdot \frac{\delta}{\delta \Phi} - dV \frac{\partial}{\partial V} \right) \Gamma_*[\Phi] - \frac{1}{2} \text{tr} \left((\dot{\mathcal{R}} + \eta \mathcal{R}) \cdot (\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \right) = 0. \quad (2.226)$$

Corresponding to (2.81) we then have

$$\Gamma_t[\Phi] \sim \Gamma_*[\Phi] - \sum_{n \geq 0} e^{-\lambda_n t} \mathcal{P}_n[\Phi] \quad \text{as } t \rightarrow \infty. \quad (2.227)$$

Although the relation between Φ and φ in general depends on t we have to first order in an expansion about the fixed point

$$\mathcal{P}_n[\Phi] = \mathcal{O}_n[\varphi] \quad \text{for} \quad \frac{\delta}{\delta \varphi} \mathcal{S}_*[\varphi] - \mathcal{Q} \cdot \varphi = -\mathcal{I} \cdot \Phi. \quad (2.228)$$

The eigenvalue equation, which is equivalent to (2.82) with (2.83), becomes

$$\begin{aligned} \lambda \mathcal{P}[\Phi] = \Delta_{\Gamma_*} \mathcal{P}[\Phi] &= \left(D^{(\delta)} \Phi \cdot \frac{\delta}{\delta \Phi} - dV \frac{\partial}{\partial V} \right) \mathcal{P}[\Phi] \\ &+ \frac{1}{2} \text{tr} \left((\dot{\mathcal{R}} + \eta \mathcal{R}) \cdot (\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot \mathcal{P}^{(2)}[\Phi] \cdot (\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \right), \end{aligned} \quad (2.229)$$

with $\mathcal{P}^{(2)}[\Phi; x, y]$ defined as in (2.208).

It is easy to check from (2.229) and (2.226) that

$$\mathcal{P}[\Phi] = 1 \cdot \Phi \quad \Rightarrow \quad \lambda = -\frac{1}{2}(d + 2 - \eta), \quad (2.230a)$$

$$\mathcal{P}[\Phi] = 1 \cdot \frac{\delta}{\delta \Phi} \Gamma_*[\Phi] \quad \Rightarrow \quad \lambda = -\frac{1}{2}(d - 2 + \eta), \quad (2.230b)$$

in agreement with (2.91a) and (2.91b).

If we consider redundant operators of the form (2.95) then using (2.228) to define an equivalent $\mathcal{P}_\Psi[\Phi]$ gives, with the results in (2.215), (2.216) and (2.218),

$$\mathcal{P}_\Psi[\Phi] = \Psi \cdot \frac{\delta}{\delta\Phi} \Gamma_*[\Phi] - \text{tr} \left((\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot \mathcal{R} \cdot \Psi^{(1)} \right), \quad (2.231)$$

for

$$\Psi^{(1)}[\Phi; x, y] = \frac{\delta}{\delta\Phi(y)} \Psi[\Phi; x], \quad \tilde{\Psi}[\Phi; p] = \tilde{\psi}[\varphi; p] \frac{1}{K(p^2)}. \quad (2.232)$$

Manifestly \mathcal{P} in (2.230b) is the redundant operator \mathcal{P}_1 .

For the zero mode operator given by (2.119) and (2.120) the corresponding operator here is also a redundant operator of the form (2.231) since

$$\mathcal{Z}[\Phi] = \mathcal{P}_\Phi[\Phi] = \Phi \cdot \frac{\delta}{\delta\Phi} \Gamma_*[\Phi] - \text{tr} \left((\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot \mathcal{R} \right). \quad (2.233)$$

To verify that this is a zero mode it is necessary to use

$$\Delta_{\Gamma_*} \mathcal{Z}[\Phi] = \Phi \cdot \frac{\delta}{\delta\Phi} E[\Phi] - \text{tr} \left((\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot E^{(2)}[\Phi] \right), \quad (2.234)$$

where E is defined in (2.226) and we require the identity

$$\begin{aligned} & -\text{tr} \left((\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot \mathcal{R} \cdot (\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot (D^{(d-\delta)} \Gamma_*^{(2)}[\Phi] + \Gamma_*^{(2)}[\Phi] \overleftarrow{D}^{(d-\delta)}) \right) \\ & = \text{tr} \left((\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot (\dot{\mathcal{R}} + \eta \mathcal{R}) \cdot (\mathcal{R} + \Gamma_*^{(2)}[\Phi])^{-1} \cdot \Gamma_*^{(2)}[\Phi] \right), \end{aligned} \quad (2.235)$$

which depends on (2.221).

If $\Gamma_t[\Phi]$ is restricted to Gaussian form so that

$$\Gamma_t[\Phi] = \frac{1}{2} \Phi \cdot \chi_t \cdot \Phi + X_t V, \quad (2.236)$$

then the solution of (2.209) gives simply

$$\tilde{\chi}_t(p) = e^{(2-\eta)t} \tilde{\chi}_0(e^{-t}p), \quad (2.237)$$

independent of \mathcal{R} . For there to be a limit as $t \rightarrow \infty$ with $\tilde{\chi}_0(p)$ analytic in p then it is necessary that $\eta = 0$, $\tilde{\chi}_0(p) = r p^2 + \dots$ and also X_0 to be chosen precisely to remove any e^{dt} terms, so that from (2.226)

$$\Gamma_*[\Phi] = \frac{1}{2} \Phi \cdot \chi_* \cdot \Phi - \frac{1}{2d} \text{tr}((\mathcal{R} + \chi_*)^{-1} \cdot \dot{\mathcal{R}}), \quad \tilde{\chi}_*(p) = r p^2, \quad (2.238)$$

where, with $\dot{\mathcal{R}}$ given by (2.225),

$$\text{tr}((\mathcal{R} + \chi_*)^{-1} \cdot \dot{\mathcal{R}}) = \frac{V}{(2\pi)^d} \int d^d p \frac{\tilde{\mathcal{R}}(p)}{\mathcal{R}(p) + r p^2}. \quad (2.239)$$

Applying the formula (2.233) to the Gaussian case gives

$$\begin{aligned}\mathcal{Z}[\Phi] &= \Phi \cdot \chi_* \cdot \Phi - \text{tr}((\mathcal{R} + \chi_*)^{-1} \cdot \mathcal{R}) \\ &= \Phi \cdot \chi_* \cdot \Phi + \frac{1}{d} \frac{V}{(2\pi)^d} \int d^d p \frac{\tilde{\mathcal{R}}(p) r p^2}{(\tilde{\mathcal{R}}(p) + r p^2)^2} = 2r \frac{\partial}{\partial r} \Gamma_*[\Phi],\end{aligned}\quad (2.240)$$

using $1 = \partial_p \cdot p/d$ and integrating by parts with the result (2.239) for $\dot{\mathcal{R}}$. In terms of the original Polchinski cut off function $K(p^2)$, and with $\eta = 0$, $\tilde{\mathcal{R}}(p) = p^2 K(p^2)/(1 - K(p^2))$, $\tilde{\mathcal{R}}(p) = 2(p^2)^2 K'(p^2)/(1 - K(p^2))^2$.

2.10 Alternate RG Equations

By considering a transform akin to that in (2.46) the Polchinski RG equation generates a new RG equation with some additional desirable features. This equation is equivalent to considering expansions of the original equation in terms of a normal ordered basis, and is connected with the approach based on using scaling fields to reduce the RG equations to a tractable set of finite equations [23].

To handle $\eta \neq 0$ it is necessary to introduce a new Green function $\hat{\mathcal{G}}$ which is defined by a modification of (2.43) by requiring

$$\left[D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi}, \hat{\mathcal{Y}} \right] = \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi}, \quad \hat{\mathcal{Y}} = \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \hat{\mathcal{G}} \cdot \frac{\delta}{\delta \varphi}. \quad (2.241)$$

The corresponding equation for $\hat{\mathcal{G}}$ is then

$$(y \cdot \partial_y + d - 2 + \eta) \hat{\mathcal{G}}(y) = -G(y) \quad \text{or} \quad (p \cdot \partial_p + 2 - \eta) \tilde{\mathcal{G}}(p) = 2K'(p^2), \quad (2.242)$$

using (2.30). This has a solution

$$\tilde{\mathcal{G}}(p) = \frac{\hat{K}(p^2)}{p^2}, \quad \hat{K}(p^2) = -(p^2)^{\frac{1}{2}\eta} \int_{p^2}^{\infty} dx K'(x) x^{-\frac{1}{2}\eta}, \quad (2.243)$$

where $\hat{K}(p^2)$ has been required to vanish for large p^2 . Clearly

$$\hat{K}(p^2) \underset{p \rightarrow 0}{\sim} \hat{C}_\eta (p^2)^{\frac{1}{2}\eta}, \quad \hat{C}_\eta = - \int_0^{\infty} dx K'(x) x^{-\frac{1}{2}\eta}, \quad (2.244)$$

or equivalently

$$\hat{\mathcal{G}}(y) \underset{y \rightarrow \infty}{\sim} \frac{k}{(y^2)^{\frac{1}{2}(d-2+\eta)}}, \quad (2.245)$$

where

$$C_\eta = (d - 2 + \eta) S_d k = \frac{2^\eta \Gamma(\frac{1}{2}d + \frac{1}{2}\eta)}{\Gamma(\frac{1}{2}d) \Gamma(1 - \frac{1}{2}\eta)} \hat{C}_\eta, \quad S_d = \frac{2\pi^{\frac{1}{2}d}}{\Gamma(\frac{1}{2}d)}. \quad (2.246)$$

For $\eta = 0$ it is easy to see that $\hat{\mathcal{G}}(y) = \mathcal{G}(y)$ and $\hat{\mathcal{Y}} = \mathcal{Y}$ with $C_0 = \hat{C}_0 = 1$ as a consequence of $K(0) = 1$, and the asymptotic form is the same as given by (2.156) with (2.75). For general η if $K(p^2)$ falls off faster than any power then from (2.243)

$$\hat{K}(p^2) \sim K(p^2) \quad \text{as } p^2 \rightarrow \infty. \quad (2.247)$$

Using (2.241) (2.24) can be recast in the form of the functional differential equation,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) e^{\hat{\mathcal{Y}}} \mathcal{S}_t[\varphi] \\ &= e^{\hat{\mathcal{Y}}} \frac{1}{2} \left(\frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi] - \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi - \text{tr}(1)) \right). \end{aligned} \quad (2.248)$$

Defining

$$\dot{\mathcal{S}}_t[\varphi] = e^{\hat{\mathcal{Y}}} \mathcal{S}_t[\varphi], \quad (2.249)$$

then, using $e^{a \frac{d^2}{dz^2}} (f(z) g(z)) = e^{2a \frac{\partial^2}{\partial z \partial z'}} (e^{a \frac{d^2}{dz^2}} f(z) e^{a \frac{d^2}{dz'^2}} g(z')) \Big|_{z'=z}$,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - dV \frac{\partial}{\partial V} \right) \dot{\mathcal{S}}_t[\varphi] + \frac{1}{2} \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi + \text{tr}(\hat{\mathcal{G}} \cdot \mathcal{G}^{-1} - 1)) \\ &= \exp \left(\frac{\delta}{\delta \varphi} \cdot \hat{\mathcal{G}} \cdot \frac{\delta}{\delta \varphi'} \right) \frac{1}{2} \frac{\delta \dot{\mathcal{S}}_t[\varphi]}{\delta \varphi} \cdot G \cdot \frac{\delta \dot{\mathcal{S}}_t[\varphi']}{\delta \varphi'} \Big|_{\varphi'=\varphi} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x d^d x' G(y) \prod_{r=1}^n d^d x_r d^d x'_r \hat{\mathcal{G}}(y_r) \\ & \quad \times \frac{\delta^{n+1} \dot{\mathcal{S}}_t[\varphi]}{\delta \varphi_i(x) \delta \varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n)} \frac{\delta^{n+1} \dot{\mathcal{S}}_t[\varphi]}{\delta \varphi_i(x') \delta \varphi_{i_1}(x'_1) \dots \delta \varphi_{i_n}(x'_n)}, \end{aligned} \quad (2.250)$$

where $y_r = x_r - x'_r$. The transformation $\mathcal{S}_t \rightarrow \dot{\mathcal{S}}_t$ is tantamount to expressing \mathcal{S}_t in a normal ordered basis. If $\dot{\mathcal{S}}_t[\varphi]$ is expanded in a basis of monomials $\mathcal{P}_n[\varphi]$ then $\mathcal{S}_t[\varphi] = e^{-\hat{\mathcal{Y}}} \dot{\mathcal{S}}_t[\varphi]$ has a corresponding expansion in terms of the normal ordered monomials $\mathcal{N}_{\hat{\mathcal{G}}}(\mathcal{P}_n[\varphi]) = e^{-\hat{\mathcal{Y}}} \mathcal{P}_n[\varphi]$, as in the definition (2.41), for the two point function $\hat{\mathcal{G}}$.

For $\eta = 0$ and $\hat{\mathcal{G}} = \mathcal{G}$ an equation essentially identical with (2.250) was used as a starting point by Wiczerkowski and Salmhofer [24, 25] in proofs of perturbative renormalisation for scalar field theories.

3 Derivative Expansion

Since the RG flow equations for \mathcal{S}_t ensure that it remains essentially local it is natural to consider an expansion where $\mathcal{S}_t[\varphi]$ is an entirely local functional $\int d^d x \mathcal{L}(\varphi, \partial \varphi, \partial \partial \varphi, \dots)$, with the expansion parameter the total number of derivatives. Although such a derivative expansion has often been used as an approximation to the exact RG functional equations [26, 27, 20, 28, 29, 30] the resulting differential equations, at least beyond lowest order, depend on the detailed form of the cut off function and reliability of any results, unless

some optimisation strategy is used, can be uncertain. It is also not obvious how to maintain consistency with perturbative results beyond lowest order.

It is also useful to consider an extension to a N -component scalar field $\phi_i \rightarrow \varphi_i$, assuming now \cdot includes contraction of indices where appropriate. The quadratic term in (2.20) $\frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \varphi$, which includes the cut off function K , is invariant under $O(N)$ symmetry but this may be reduced depending on the form of the initial $\mathcal{S}_0[\varphi]$ in the RG flow equations. For the present discussion we start from the Polchinski equation (2.24) which includes η . In the multi-component case this becomes in general a matrix but for simplicity we assume $\eta_{ij} = \eta \delta_{ij}$, as would be required by $O(N)$ symmetry.

Usually the derivative expansion is applied directly to $\mathcal{S}_t[\varphi]$. Here we discuss an alternative form of derivative expansion, different from the standard approach, which is in terms of $\mathring{\mathcal{S}}_t$, defined in (2.249), by writing

$$\mathring{\mathcal{S}}_t[\varphi] = \int d^d x \left(\mathring{V}(\varphi) + \frac{1}{2} \partial^\mu \varphi_j \partial_\mu \varphi_k \mathring{Z}_{jk}(\varphi) + \dots \right). \quad (3.1)$$

It is easy to obtain

$$\begin{aligned} & \left(D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - d V \frac{\partial}{\partial V} \right) \mathring{\mathcal{S}}_t[\varphi] \\ &= \int d^d x \left(-d \mathring{V}(\varphi) + \delta \varphi_i \frac{\partial}{\partial \varphi_i} \mathring{V}(\varphi) \right. \\ & \quad \left. + \frac{1}{2} \partial^\mu \varphi_j \partial_\mu \varphi_k \left(\eta \mathring{Z}_{jk}(\varphi) + \delta \varphi_i \frac{\partial}{\partial \varphi_i} \mathring{Z}_{jk}(\varphi) \right) + \dots \right). \end{aligned} \quad (3.2)$$

If (3.1) is inserted on the right hand side of (2.250) the functional derivatives generate δ -functions ensuring that the integrals over x_r, x'_r all become trivial to evaluate so there remain integrals just over x, x' . The resulting expression corresponds to contributions from all two point Feynman graphs. The leading terms, with up to two derivatives, are then, after judicious integrations by parts,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} - d V \frac{\partial}{\partial V} \right) \mathring{\mathcal{S}}_t[\varphi] + \frac{1}{2} \eta (\varphi \cdot \mathcal{G}^{-1} \cdot \varphi + \text{tr}(\hat{\mathcal{G}} \cdot \mathcal{G}^{-1} - 1)) \\ & \sim \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x d^d x' \left\{ \mathring{V}_{i_1 \dots i_n}(\varphi) \mathring{V}_{i_1 \dots i_n}(\varphi') G(y) \hat{\mathcal{G}}(y)^n \right. \\ & \quad + \partial^\mu \varphi_j \partial_\mu \varphi_k \mathring{Z}_{jk, i_1 \dots i_n}(\varphi) \mathring{V}_{i_1 \dots i_n}(\varphi') G(y) \hat{\mathcal{G}}(y)^n \\ & \quad + \partial^\mu \varphi_j \partial_\mu \varphi'_k \left(2 \mathring{Z}_{ji, i_1 \dots i_n}(\varphi) - \mathring{Z}_{ii_1, j i_2 \dots i_n}(\varphi) \right) \mathring{V}_{ki i_1 \dots i_n}(\varphi') G(y) \hat{\mathcal{G}}(y)^n \\ & \quad \left. - \mathring{Z}_{jk, i_1 \dots i_{n-1}}(\varphi) \mathring{V}_{jk i_1 \dots i_{n-1}}(\varphi') \left(\partial^2 G(y) \hat{\mathcal{G}}(y)^n + n G(y) \partial^2 g(y) \hat{\mathcal{G}}(y)^{n-1} \right) \right\}, \end{aligned} \quad (3.3)$$

where in the integral $\varphi' = \varphi(x')$ while $\varphi = \varphi(x)$, as before $y = x - x'$, and we adopt the notation $\mathring{V}_{i_1 \dots i_n}(\varphi) = \partial_{i_1} \partial_{i_2} \dots \partial_{i_n} \mathring{V}(\varphi)$, and similarly for derivatives of $\mathring{Z}_{jk}(\varphi)$.

To obtain tractable closed equations which may be solved it is necessary to project the right hand side of (3.3) onto local expressions of the same form as feature in the derivative

expansion. This is achieved by assuming that the products of $G(y)$ and $\hat{\mathcal{G}}(y)$ that feature in (3.3) can be expanded in the form

$$\begin{aligned} G(y) \hat{\mathcal{G}}(y)^n &\sim -c_n \delta^d(y) - c'_n \partial^2 \delta^d(y) + \dots, \\ \partial^2 G(y) \hat{\mathcal{G}}(y)^n + n G(y) \partial^2 \hat{\mathcal{G}}(y) \hat{\mathcal{G}}(y)^{n-1} &\sim d_n \delta^d(y) + \dots \end{aligned} \quad (3.4)$$

In general c_n, c'_n, d_n , for arbitrary n and dimension d , depend on the detailed form of the cut off function. However some results are independent of the precise form of the cut off and depend only on the form of $\hat{\mathcal{G}}(y)$ for large y as exhibited in (2.245). To demonstrate this we consider the integrals

$$\begin{aligned} - \int d^d y (y^2)^r G(y) \hat{\mathcal{G}}(y)^n &= \frac{1}{n+1} \int d^d y (y^2)^r (y \cdot \partial_y + (n+1)(d-2+\eta)) \hat{\mathcal{G}}(y)^{n+1} \\ &= \frac{1}{n+1} \int d^d y \partial_\mu \left(y^\mu \frac{k^{n+1}}{(y^2)^{\frac{1}{2}d}} \right) = \frac{C_\eta}{d+2r} k^n \quad \text{if } (n+1)(d-2+\eta) - 2r = d, \end{aligned} \quad (3.5)$$

using (2.242) and where C_η is given by (2.246). For $\eta = 0$, and $C_0 = 1$, these results are just the coefficients of the logarithmic divergencies in the associated two point Feynman graphs in appropriate dimensions depending on n . Using (3.5) it is then easy to see that for $n = 1, 2, \dots$,

$$c_n|_{d=(2-\eta)(n+1)/n} = \frac{C_\eta}{d} k^n, \quad c'_n|_{d=((2-\eta)(n+1)+2)/n} = \frac{C_\eta}{2d(d+2)} k^n. \quad (3.6)$$

There are no such comparable results for d_n and furthermore it is evident that $d_0 = 0$. In consequence we assume the simplest expressions for c_n, c'_n, d_n consistent with this, and which interpolate (3.6) for all n, d including $n = 0$, and take henceforth

$$c_n = \frac{C_\eta}{d} k^n, \quad c'_n = \frac{C_\eta}{2d(d+2)} k^n, \quad d_n = 0. \quad (3.7)$$

If (3.4), with (3.7), is inserted in (3.3) then, along with (3.2), we obtain equations for the function \mathring{V} ,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - d + \delta \varphi_i \frac{\partial}{\partial \varphi_i} \right) \mathring{V}(\varphi) &= - \frac{C_\eta}{2d} \sum_{n=0}^{\infty} \frac{k^n}{n!} \mathring{V}_{i_1 \dots i_{n+1}}(\varphi) \mathring{V}_{i_1 \dots i_{n+1}}(\varphi) \\ &= - \frac{C_\eta}{2d} e^{k \frac{\partial^2}{\partial \varphi_j \partial \varphi'_j}} \frac{\partial}{\partial \varphi_i} \mathring{V}(\varphi) \frac{\partial}{\partial \varphi'_i} \mathring{V}(\varphi') \Big|_{\varphi'=\varphi}, \end{aligned} \quad (3.8)$$

and also \mathring{Z}_{jk} ,

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \eta + \delta \varphi_i \frac{\partial}{\partial \varphi_i} \right) \mathring{Z}_{jk}(\varphi) + \eta \delta_{jk} \\ &= - \frac{C_\eta}{d} \sum_{n=0}^{\infty} \frac{k^n}{n!} \left(\mathring{V}_{i_1 \dots i_{n+1}}(\varphi) \mathring{Z}_{jk, i_1 \dots i_{n+1}}(\varphi) \right. \\ &\quad \left. + \mathring{V}_{i_1 \dots i_{n+1}(j)}(\varphi) (2 \mathring{Z}_{k) i_1, i_2 \dots i_{n+1}}(\varphi) - \mathring{Z}_{i_1 i_2, k) i_3 \dots i_{n+1}}(\varphi) \right) \\ &\quad + \frac{C_\eta}{2d(d+2)} \sum_{n=0}^{\infty} \frac{k^n}{n!} \mathring{V}_{j i_1 \dots i_{n+1}}(\varphi) \mathring{V}_{k i_1 \dots i_{n+1}}(\varphi). \end{aligned} \quad (3.9)$$

For a single component φ , $N = 1$, this can be written in the form

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \eta + \delta \varphi \frac{\partial}{\partial \varphi} \right) \dot{Z}(\varphi) + \eta \\ &= -e^{k \frac{\partial^2}{\partial \varphi \partial \varphi'}} \frac{C_\eta}{d} \left(\frac{\partial}{\partial \varphi} \dot{V}(\varphi) \frac{\partial}{\partial \varphi'} \dot{Z}(\varphi') + \frac{\partial^2}{\partial \varphi^2} \dot{V}(\varphi) \dot{Z}(\varphi') \right. \\ & \quad \left. - \frac{1}{2(d+2)} \frac{\partial^2}{\partial \varphi^2} \dot{V}(\varphi) \frac{\partial^2}{\partial \varphi'^2} \dot{V}(\varphi') \right) \Big|_{\varphi'=\varphi}, \end{aligned} \quad (3.10)$$

Assuming now the relevant solutions of (3.8) satisfy

$$\dot{V}(\varphi) = \frac{2d\omega^2}{C_\eta} e^{\frac{1}{2}k \frac{\partial^2}{\partial \varphi_i \partial \varphi_i}} V(\varphi/\omega), \quad \omega^2 = \frac{1}{2}(d-2+\eta)k, \quad (3.11)$$

then (3.8) reduces to

$$\left(\frac{\partial}{\partial t} - d + \delta \varphi_i \frac{\partial}{\partial \varphi_i} - \frac{\partial^2}{\partial \varphi_i \partial \varphi_i} \right) V(\varphi) = - \left(\frac{\partial}{\partial \varphi_i} V(\varphi) \right)^2. \quad (3.12)$$

Moreover for a single component field if

$$\dot{Z}(\varphi) = e^{\frac{1}{2}k \frac{\partial^2}{\partial \varphi^2}} Z(\varphi/\omega), \quad (3.13)$$

then (3.10) becomes

$$\left(\frac{\partial}{\partial t} + \eta + \Delta_V + 2V''(\varphi) \right) Z(\varphi) = -\eta + \frac{2d}{(d+2)C_\eta} V''(\varphi)^2. \quad (3.14)$$

where

$$\Delta_V = -\frac{\partial^2}{\partial \varphi^2} + \delta \varphi \frac{\partial}{\partial \varphi} + 2V'(\varphi) \frac{\partial}{\partial \varphi}. \quad (3.15)$$

Solutions of (3.12) and (3.10) clearly generate solutions of (3.8) and (3.10) using (3.11) (in general (3.11) cannot be inverted but V, Z are defined by the requirement of satisfying (3.12) and (3.14)). It is not clear how to reduce the more general equation (3.9) to a similar form as in (3.14) since it is not possible in general to extract a factor $e^{k \frac{\partial^2}{\partial \varphi_j \partial \varphi_j}}$ on the right hand side of (3.9), as was done in (3.8).

The procedure adopted here in obtaining a derivative expansion is analogous to that used in the scaling field approach but there \dot{V}, \dot{Z} are expanded in a basis of monomials in φ and it is then possible to use the more general form (3.4) for the products of propagators appearing in the expansion (3.3).

3.1 Extension of the Local Potential Approximation

For a single component field at a fixed point (3.12) becomes

$$-d V_*(\varphi) + \delta \varphi V'_*(\varphi) - V''_*(\varphi) = -V'_*(\varphi)^2. \quad (3.16)$$

It is furthermore consistent to require $\eta = O(V_*^2)$ in which case (3.14) may be simplified at the fixed point determined by (3.16) by restricting, in a expansion in powers of V_* , only to terms up to $O(Z_* V_*)$ to the form

$$(\Delta_{V_*} + 2V_*''(\phi))Z_*(\phi) = -\eta + \frac{2d}{d+2} V_*''(\phi)^2, \quad (3.17)$$

taking also $C_\eta \rightarrow 1$. Combining (3.17) and (3.16) is a minimal extension of the local potential approximation to include a non zero anomalous dimension η which was by discussed Osborn and Twigg [31]. The associated eigenvalue equation for critical exponents becomes

$$\begin{pmatrix} \Delta_{V_*} - d & 0 \\ 2\frac{d}{d\phi}Z_*(\phi)\frac{d}{d\phi} - \frac{4d}{d+2}V_*''(\phi)\frac{d^2}{d\phi^2} & \Delta_{V_*} + 2V_*''(\phi) \end{pmatrix} \begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = \lambda \begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix}, \quad (3.18)$$

which is easy to analyse numerically [31] since it can be reduced to solving $(\Delta_{V_*} - d)f = \lambda f$ and $(\Delta_{V_*} + 2V_*'')g = \lambda g$.

The virtue of (3.16) and (3.17) is that the eigenvalue problem (3.18) has exact eigenfunctions and eigenvalues which match those of the RG equations as discussed in subsections 2.4 and 2.5. Corresponding to (2.91a), (2.91b) in (3.18)

$$\begin{aligned} f_\varphi(\varphi) &= \varphi - \frac{2}{2-\eta} V_*'(\varphi), & g_\varphi(\varphi) &= -\frac{2}{2-\eta} Z_*'(\varphi), & \lambda_\varphi &= -\frac{1}{2}(d+2-\eta), \\ f_r(\varphi) &= V_*'(\varphi), & g_r(\varphi) &= Z_*'(\varphi), & \lambda_r &= -\frac{1}{2}(d-2+\eta). \end{aligned} \quad (3.19)$$

Corresponding to the zero mode

$$f_Z(\varphi) = 0, \quad g_Z(\varphi) = 1 - \frac{2}{2-\eta} V_*''(\varphi). \quad (3.20)$$

As a consequence of this zero mode solution there exist non trivial solutions of the homogeneous equation (3.17) so that $\Delta_{V_*} + 2V_*''$ is not invertible. The existence of a solution for Z_* imposes an eigenvalue condition on η , since the right side of (3.17) must be orthogonal to $g_Z(\varphi)$ with respect to a suitable scalar product, constructed in [31], for which Δ_{V_*} is hermitian. For η so determined there is then a line of equivalent fixed points $Z_*(\varphi) \sim Z_*(\varphi) + c g_Z(\varphi)$ for any c .

These results may illustrated by an epsilon expansion close to the multi-critical points arising when $\varphi^{2(n+1)}$, $n = 1, 2, \dots$, becomes a marginal operator which occurs, when $\eta = 0$, for $(n+1)\delta_0 = d$ or $d = d_n = 2(n+1)/n$. Using $e^{\frac{1}{2\delta_0}\frac{d^2}{d\varphi^2}}(-\frac{d^2}{d\varphi^2} + \delta_0\varphi\frac{d}{d\varphi}) = \delta_0\varphi\frac{d}{d\varphi}e^{\frac{1}{2\delta_0}\frac{d^2}{d\varphi^2}}$ then approximating (3.16) gives

$$d = d_n - \epsilon, \quad e^{\frac{1}{2\delta_0}\frac{d^2}{d\varphi^2}} V_*(\varphi) = g_n \epsilon \varphi^{2(n+1)} + O(\epsilon^2 \varphi^{2p}, p \neq n+1), \quad (3.21)$$

where, for $\epsilon > 0$,

$$g_n = n n! \delta_0^n \left(\frac{n!}{2(2n+1)!} \right)^2. \quad (3.22)$$

The corresponding solutions for $V_*(\varphi)$ are expressible in terms of Hermite polynomials using the identity

$$e^{-\frac{1}{4} \frac{d^2}{dx^2}} (2x)^r = H_r(x). \quad (3.23)$$

At any fixed order in the ϵ -expansion there is a finite sum of H_r .

The condition for (3.17) then to have solutions to leading order becomes

$$e^{\frac{1}{2\delta_0} \frac{d^2}{d\varphi^2}} \left(-\eta + \frac{2d_n}{d_n + 2} V_*''(\varphi)^2 \right) = O(\varphi^{2p}, p \neq 0), \quad (3.24)$$

giving [3]

$$\eta = 8(n+1)^3 \frac{(2n+1)!}{\delta_0^{2n}} g_n^2 \epsilon^2 = 4n^2 \left(\frac{(n+1)!^2}{(2n+2)!} \right)^3 \epsilon^2. \quad (3.25)$$

With this result for η (3.17) has the solution $Z_*(\varphi) = c(1 - V_*''(\varphi)) + O(\epsilon^2)$ for arbitrary c .

4 Supersymmetric Example

As a small example of the derivative expansion we consider a three dimensional field theory with $\mathcal{N} = 2$ supersymmetry. Applications of RG flow equations to supersymmetric theories have been described in [32, 33, 34, 35]. For three dimensional $\mathcal{N} = 2$ theories there are chiral superfields which may have a holomorphic superpotential, like $\mathcal{N} = 1$ Wess Zumino theories in four dimensions from which they may be obtained by reduction. Unlike the four dimensional theories [34] non trivial IR fixed points in the absence of any gauge fields are not excluded.

As usual with supersymmetry it is convenient to adopt a spinorial notation using the result that in three dimensions the gamma matrices may be realised in terms of symmetric real 2×2 matrices

$$(\sigma_a)_{\alpha\beta} = (\sigma_a)_{\beta\alpha}, \quad (\tilde{\sigma}_a)^{\alpha\beta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} (\sigma_a)_{\gamma\delta}, \quad (4.1)$$

with $\alpha, \beta = 1, 2$ and

$$\sigma_a \tilde{\sigma}_b + \sigma_b \tilde{\sigma}_a = -2\eta_{ab} I, \quad (4.2)$$

with η_{ab} the 3-dimensional Minkowski metric with signature $(-1, 1, 1)$ and I the identity matrix. Any 3-vector x^a is then equivalent to a symmetric 2×2 matrix using the σ -matrices in (4.1),

$$x^a \rightarrow x_{\alpha\beta} = (x^a \sigma_a)_{\alpha\beta}, \quad \tilde{x}^{\alpha\beta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} x_{\gamma\delta}, \quad (4.3)$$

so that $x\tilde{x} = -x^2 I$. We also define

$$\partial_{\alpha\beta} = (\sigma^a \partial_a)_{\alpha\beta}, \quad \tilde{\partial}^{\alpha\beta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \partial_{\gamma\delta}, \quad (4.4)$$

so that

$$\partial_{\alpha\beta} \tilde{x}^{\gamma\delta} = -\delta_\alpha^\gamma \delta_\beta^\delta - \delta_\alpha^\delta \delta_\beta^\gamma. \quad (4.5)$$

For $\mathcal{N} = 2$ superfields there are additional anti-commuting Grassmannian coordinates $\theta^\alpha, \bar{\theta}^\alpha$. The associated covariant derivatives are D_α, \bar{D}_α , where $D_\alpha \theta^\beta = \delta_\alpha^\beta$ and $\bar{D}_\alpha \bar{\theta}^\beta = \delta_\alpha^\beta$, satisfy

$$\{D_\alpha, \bar{D}_\beta\} = -2i \partial_{\alpha\beta}. \quad (4.6)$$

D_α, \bar{D}_α anti-commute with the generators of super-translations. The full $\mathcal{N} = 2$ superspace $\mathbb{M}^{3|4}$ with coordinates $(x, \theta, \bar{\theta})$ contains the invariant chiral superspaces $\mathbb{M}^{3|2}$ and $\bar{\mathbb{M}}^{3|2}$, defined in terms the chiral coordinates (x_+, θ) and $(x_-, \bar{\theta})$, where $x_\pm^{\alpha\beta}$ are required to satisfy

$$\bar{D}_\gamma x_+^{\alpha\beta} = 0, \quad D_\gamma x_-^{\alpha\beta} = 0. \quad (4.7)$$

Chiral superfields are defined on $\mathbb{M}^{3|2}$ and their anti-chiral conjugates on $\bar{\mathbb{M}}^{3|2}$ so that

$$\bar{D}_\alpha \phi = 0 \Rightarrow \phi(x_+, \theta), \quad D_\alpha \bar{\phi} = 0 \Rightarrow \bar{\phi}(x_-, \bar{\theta}). \quad (4.8)$$

For two points labelled by $(x, \theta, \bar{\theta})$ and $(x', \theta', \bar{\theta}')$ there is a supertranslation invariant generalisation y of the interval $x - x'$ given by

$$\begin{aligned} y^{\alpha\beta} &= \tilde{x}_+^{\alpha\beta} - \tilde{x}'^{\alpha\beta} - 4i \theta^{(\alpha} \bar{\theta}'^{\beta)} \\ &= \tilde{x}^{\alpha\beta} - \tilde{x}'^{\alpha\beta} + 2i \theta^{(\alpha} (\bar{\theta}^{\beta)} - \bar{\theta}'^{\beta)}) - 2i (\theta^{(\alpha} - \theta'^{(\alpha}) \bar{\theta}'^{\beta)}). \end{aligned} \quad (4.9)$$

It is easy to see that this satisfies $\bar{D}_\gamma y^{\alpha\beta} = D'_\gamma y^{\alpha\beta} = 0$ as y is a function on $\mathbb{M}^{3|2} \times \bar{\mathbb{M}}^{3|2}$.

For chiral or anti-chiral fields we may extend (2.3) to

$$\phi \cdot \psi = \int d^3x d^2\theta \phi(x, \theta) \psi(x, \theta), \quad \bar{\phi} \cdot \bar{\psi} = \int d^3x d^2\bar{\theta} \bar{\phi}(x, \bar{\theta}) \bar{\psi}(x, \bar{\theta}), \quad (4.10)$$

and the associated functional derivatives are correspondingly defined so that

$$\begin{aligned} \frac{\delta}{\delta\phi(x, \theta)} \phi(x', \theta') &= \delta^3(x - x') (\theta - \theta')^2, \quad \theta^2 = \varepsilon_{\alpha\beta} \theta^\alpha \theta^\beta, \\ \frac{\delta}{\delta\bar{\phi}(x, \bar{\theta})} \bar{\phi}(x', \bar{\theta}') &= \delta^3(x - x') (\bar{\theta} - \bar{\theta}')^2, \quad \bar{\theta}^2 = \varepsilon_{\alpha\beta} \bar{\theta}^\alpha \bar{\theta}^\beta, \end{aligned} \quad (4.11)$$

where $(\theta - \theta')^2, (\bar{\theta} - \bar{\theta}')^2$ play the role of Grassmannian delta functions, the integrations over $\theta, \bar{\theta}$ being normalised so that $\int d^2\theta \theta^2 = \int d^2\bar{\theta} \bar{\theta}^2 = 1$. In order to write RG equations analogous to those for simple scalar fields we define bilinear expressions involving a chiral field ϕ and an anti-chiral field $\bar{\phi}$, extending (2.7), by

$$\phi \cdot G \cdot \bar{\phi} = \int d^3x_+ d^2\theta d^3x'_- d^2\bar{\theta}' \phi(x_+, \theta) G(y) \bar{\phi}(x'_-, \bar{\theta}'), \quad (4.12)$$

for y defined by (4.9). Using, with this definition,

$$G(y) = \frac{1}{16} \bar{D}^2 D'^2 (G(x - x') (\theta - \theta')^2 (\bar{\theta} - \bar{\theta}')^2). \quad (4.13)$$

the expression (4.12) may be elevated from integrals over $\mathbb{M}^{3|2}$ and $\bar{\mathbb{M}}^{3|2}$ to integrals over the full superspace $\mathbb{M}^{3|4}$ and $\bar{\mathbb{M}}^{3|4}$ by letting

$$d^3x_+ d^2\theta (-\tfrac{1}{4} \bar{D}^2) \rightarrow d^3x d^2\theta d^2\bar{\theta}, \quad d^3x'_- d^2\bar{\theta}' (-\tfrac{1}{4} D'^2) \rightarrow d^3x' d^2\theta' d^2\bar{\theta}', \quad (4.14)$$

and hence

$$\phi \cdot G \cdot \bar{\phi} = \int d^3x d^2\theta d^2\bar{\theta} d^3x' d^2\theta' d^2\bar{\theta}' \phi(x_+, \theta) G(x - x') (\theta - \theta')^2 (\bar{\theta} - \bar{\theta}')^2 \bar{\phi}(x'_-, \bar{\theta}'). \quad (4.15)$$

For the identity I , where $I(y) = \delta^3(y)$, then

$$S_0[\phi, \bar{\phi}] = \phi \cdot I \cdot \bar{\phi} = \int d^3x d^2\theta d^2\bar{\theta} \phi(x_+, \theta) \bar{\phi}(x_-, \bar{\theta}). \quad (4.16)$$

is just the standard free kinetic term.

As before in (2.9) it is convenient to rescale by using the cut off Λ to dimensionless variables so that

$$\phi(x, \theta) = \Lambda^{\frac{1}{2}} \varphi(x\Lambda, \theta\Lambda^{\frac{1}{2}}), \quad \bar{\phi}(x, \bar{\theta}) = \Lambda^{\frac{1}{2}} \bar{\varphi}(x\Lambda, \bar{\theta}\Lambda^{\frac{1}{2}}). \quad (4.17)$$

Just as for scalar field theories RG flow equations analogous to (2.24) may be written in a similar form for $\mathcal{S}_t[\varphi, \bar{\varphi}]$ where

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} + D^{(\delta)} \bar{\varphi} \cdot \frac{\delta}{\delta \bar{\varphi}} \right) \mathcal{S}_t[\varphi, \bar{\varphi}] \\ &= \frac{\delta}{\delta \varphi} \mathcal{S}_t[\varphi, \bar{\varphi}] \cdot G \cdot \frac{\delta}{\delta \bar{\varphi}} \mathcal{S}_t[\varphi, \bar{\varphi}] - \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \bar{\varphi}} \mathcal{S}_t[\varphi, \bar{\varphi}] - \eta \varphi \cdot \mathcal{G}^{-1} \cdot \bar{\varphi}, \end{aligned} \quad (4.18)$$

for, arising from the rescaling in (4.17),

$$\begin{aligned} D^{(\delta)} \varphi(x, \theta) &= (x \cdot \partial_x + \tfrac{1}{2} \theta^\alpha \partial_\alpha + \delta) \varphi(x, \theta), \\ D^{(\delta)} \bar{\varphi}(x, \bar{\theta}) &= (x \cdot \partial_x + \tfrac{1}{2} \bar{\theta}^\alpha \bar{\partial}_\alpha + \delta) \bar{\varphi}(x, \bar{\theta}), \end{aligned} \quad (4.19)$$

with $\partial_\alpha = \partial/\partial\theta^\alpha$, $\bar{\partial}_\alpha = \partial/\partial\bar{\theta}^\alpha$. In this case

$$\delta = \tfrac{1}{2} + \tfrac{1}{2}\eta, \quad (4.20)$$

with η the anomalous dimension. From the definition (4.9)

$$D^{(\delta)} F(y) + F(y) \overleftarrow{D}^{(\delta)} = (y \cdot \partial_y + 2\delta) F(y). \quad (4.21)$$

In this case there are no V terms as the superspace volume vanishes. In (4.18) we have introduced η representing a one parameter arbitrariness in the choice of RG flow equations. As in the earlier discussion this is expected to be constrained for $\mathcal{S}_t[\varphi, \bar{\varphi}]$ to have a well defined limit, realising a non trivial IR fixed point, as $t \rightarrow \infty$. We also require that G, \mathcal{G}^{-1} satisfy the corresponding equation to (2.23) which, as a consequence of (4.21), leads to the same relation as in (2.26) for $\delta_0 = \frac{1}{2}$, $d = 3$.

The previous discussion of a derivative expansion can be easily adapted to the present case. Instead of (2.241) we require

$$\left[D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} + D^{(\delta)} \bar{\varphi} \cdot \frac{\delta}{\delta \bar{\varphi}}, \hat{\mathcal{Y}} \right] = \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \bar{\varphi}}, \quad (4.22)$$

which gives now

$$\hat{\mathcal{Y}} = \frac{\delta}{\delta\varphi} \cdot \hat{\mathcal{G}} \cdot \frac{\delta}{\delta\bar{\varphi}}. \quad (4.23)$$

with $\hat{\mathcal{G}}(y)$ satisfying (2.26) as before. Following (2.249) so that $\mathring{S}_t[\varphi, \bar{\varphi}] = e^{\hat{\mathcal{Y}}} S_t[\varphi, \bar{\varphi}]$ then (2.250) becomes in this case

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} + D^{(\delta)}\bar{\varphi} \cdot \frac{\delta}{\delta\bar{\varphi}} \right) \mathring{S}_t[\varphi, \bar{\varphi}] + \eta \varphi \cdot \mathcal{G}^{-1} \cdot \bar{\varphi} \\ &= \exp \left(\frac{\delta}{\delta\varphi} \cdot \hat{\mathcal{G}} \cdot \frac{\delta}{\delta\bar{\varphi}'} + \frac{\delta}{\delta\bar{\varphi}'} \cdot \hat{\mathcal{G}} \cdot \frac{\delta}{\delta\varphi} \right) \frac{\delta}{\delta\varphi} \mathring{S}_t[\varphi, \bar{\varphi}] \cdot G \cdot \frac{\delta}{\delta\bar{\varphi}'} \mathring{S}_t[\varphi', \bar{\varphi}'] \Big|_{\varphi'=\varphi, \bar{\varphi}'=\bar{\varphi}}. \end{aligned} \quad (4.24)$$

For a derivative expansion we may then write

$$\begin{aligned} \mathring{S}_t[\varphi, \bar{\varphi}] &= \int d^3x d^2\theta \mathring{W}(\varphi) + \int d^3x d^2\bar{\theta} \mathring{\bar{W}}(\bar{\varphi}) \\ &+ \int d^3x d^2\theta d^2\bar{\theta} \left[\mathring{K}(\varphi, \bar{\varphi}) + D^2\varphi \mathring{L}(\varphi, \bar{\varphi}) + \bar{D}^2\bar{\varphi} \mathring{\bar{L}}(\varphi, \bar{\varphi}) + \dots \right], \end{aligned} \quad (4.25)$$

where only the superpotential terms $\mathring{W}, \mathring{\bar{W}}$ involve just integrals over the chiral superspaces \mathcal{S}_{\pm} and in the second line $\varphi(x_+, \theta), \bar{\varphi}(x_-, \bar{\theta})$. In general $\mathring{K}(\varphi, \bar{\varphi})$ is a Kähler potential such that

$$\mathring{K}(\varphi, \bar{\varphi}) \sim \mathring{K}(\varphi, \bar{\varphi}) + f(\varphi) + \bar{f}(\bar{\varphi}). \quad (4.26)$$

The separation in (4.25) between local superpotential terms involving integrals over $\mathbb{M}^{3|2}$, $\bar{\mathbb{M}}^{3|2}$ and the remainder which contains only full superspace integrations is valid beyond any derivative expansion and is at the root of the non renormalisation theorems for supersymmetric quantum field theories.

Inserting (4.25) on the left hand side of (4.24) we may use

$$\begin{aligned} & \left(D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} + D^{(\delta)}\bar{\varphi} \cdot \frac{\delta}{\delta\bar{\varphi}} \right) \mathring{S}_t[\varphi, \bar{\varphi}] \\ &= \int d^3x d^2\theta \left(-2\mathring{W}(\varphi) + \delta\varphi \frac{\partial}{\partial\varphi} \mathring{W}(\varphi) \right) + \int d^3x d^2\bar{\theta} \left(-2\mathring{\bar{W}}(\bar{\varphi}) + \delta\bar{\varphi} \frac{\partial}{\partial\bar{\varphi}} \mathring{\bar{W}}(\bar{\varphi}) \right) \\ &+ \int d^3x d^2\theta d^2\bar{\theta} \left[-\mathring{K}(\varphi, \bar{\varphi}) + \delta \left(\varphi \frac{\partial}{\partial\varphi} + \bar{\varphi} \frac{\partial}{\partial\bar{\varphi}} \right) \mathring{K}(\varphi, \bar{\varphi}) \right. \\ &\quad + \delta D^2\varphi \left(\mathring{L}(\varphi, \bar{\varphi}) + \left(\varphi \frac{\partial}{\partial\varphi} + \bar{\varphi} \frac{\partial}{\partial\bar{\varphi}} \right) \mathring{L}(\varphi, \bar{\varphi}) \right) \\ &\quad \left. + \delta \bar{D}^2\bar{\varphi} \left(\mathring{\bar{L}}(\varphi, \bar{\varphi}) + \left(\varphi \frac{\partial}{\partial\varphi} + \bar{\varphi} \frac{\partial}{\partial\bar{\varphi}} \right) \mathring{\bar{L}}(\varphi, \bar{\varphi}) \right) + \dots \right]. \end{aligned} \quad (4.27)$$

On the right hand side of (4.24) we may expand the exponential. As a result of (4.13), and similarly for $\mathring{g}(y)$, all integrations are over the full superspace \mathcal{S} . Consequently there are no contributions involving integrals just over $\mathbb{M}^{3|2}, \bar{\mathbb{M}}^{3|2}$ which would correspond to terms in the RG flow of the superpotentials. Hence we have, without approximation, the linear equations

$$\left(\frac{\partial}{\partial t} + \delta\varphi \frac{\partial}{\partial\varphi} - 2 \right) \mathring{W}(\varphi) = 0, \quad \left(\frac{\partial}{\partial t} + \delta\bar{\varphi} \frac{\partial}{\partial\bar{\varphi}} - 2 \right) \mathring{\bar{W}}(\bar{\varphi}) = 0. \quad (4.28)$$

The solution for any t , and η , is simple

$$\mathring{W}_t(\varphi) = e^{2t} \mathring{W}_0(e^{-\delta t} \varphi), \quad \mathring{W}_t(\bar{\varphi}) = e^{2t} \mathring{W}_0(e^{-\delta t} \bar{\varphi}). \quad (4.29)$$

As discussed earlier η is constrained by the requirement of a limit as $t \rightarrow \infty$. $\mathring{W}(\varphi)$ and $\mathring{W}(\bar{\varphi})$ are required to be holomorphic functions for any finite $\varphi, \bar{\varphi}$ so the possible limits can only be of the form

$$\mathring{W}_t(\varphi) \rightarrow \mathring{W}_*(\varphi) = g_* \varphi^n, \quad \mathring{W}_t(\bar{\varphi}) \rightarrow \mathring{W}_*(\bar{\varphi}) = \bar{g}_* \bar{\varphi}^n, \quad n = 2, 3, \dots \quad (4.30)$$

This requires

$$\eta = \frac{4}{n} - 1, \quad (4.31)$$

and also, to achieve the limit for any particular n , that the initial $\mathring{W}_0(\varphi)$ and $\mathring{W}_0(\bar{\varphi})$ are constrained to contain no terms $O(\varphi^p, \bar{\varphi}^p)$ with $p < n$. Any terms with $p > n$ are irrelevant and vanish in the limit (4.30). For unitary theories $\eta \geq 0$, or $n \leq 4$, with $\eta = 0$, when $n = 4$, corresponding to a free massless theory. The $n = 2$ case is also a massive free theory. However in three dimensions, unlike the four dimensional case, there is the possibility of a non trivial IR fixed point when $n = 3$ and $\eta = \frac{1}{3}$. Given that η is not then close to zero this fixed point is clearly non perturbative.

Higher order equations in the derivative approximation may be obtained in the same fashion as for simple scalar field theories. Using just

$$G(y) \hat{\mathcal{G}}(y)^n \sim \frac{C_\eta}{3} k^n \delta^3(y) = \frac{C_\eta}{3} k^n \frac{1}{16} \bar{D}^2 D'^2 (\delta^3(x - x') (\theta - \theta')^2 (\bar{\theta} - \bar{\theta}')^2) \quad (4.32)$$

we may obtain from (4.24) and (4.27), using (4.25) with (4.14),

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - 1 + \delta \left(\varphi \frac{\partial}{\partial \varphi} + \bar{\varphi} \frac{\partial}{\partial \bar{\varphi}} \right) \right) \mathring{K}(\varphi, \bar{\varphi}) \\ &= -\eta \varphi \bar{\varphi} + \frac{2C_\eta}{3} \sum_{n=0}^{\infty} \frac{k^n}{n!} \frac{\partial^{n+1}}{\partial \varphi^{n+1}} \mathring{W}(\varphi) \frac{\partial^{n+1}}{\partial \bar{\varphi}^{n+1}} \mathring{W}(\bar{\varphi}), \end{aligned} \quad (4.33)$$

up to contributions reflecting the freedom in (4.26). Defining

$$\begin{aligned} \mathring{K}(\varphi, \bar{\varphi}) &= \omega^2 e^{k \frac{\partial^2}{\partial \varphi \partial \bar{\varphi}}} K(\varphi/\omega, \bar{\varphi}/\omega), \quad \omega^2 = (1 + \eta)k \\ \mathring{W}(\varphi) &= A \omega^2 W(\varphi/\omega), \quad \mathring{W}(\bar{\varphi}) = A \omega^2 \bar{W}(\bar{\varphi}/\omega), \quad A^2 = \frac{3}{2C_\eta}, \end{aligned} \quad (4.34)$$

then (4.33) becomes

$$\left(\frac{\partial}{\partial t} - 1 + \Delta \right) K(\varphi, \bar{\varphi}) = -\eta \varphi \bar{\varphi} + W'(\varphi) \bar{W}'(\bar{\varphi}), \quad (4.35)$$

for

$$\Delta = -\frac{\partial^2}{\partial \varphi \partial \bar{\varphi}} + \delta \left(\varphi \frac{\partial}{\partial \varphi} + \bar{\varphi} \frac{\partial}{\partial \bar{\varphi}} \right). \quad (4.36)$$

To eliminate the freedom in (4.26) we may define

$$Z(\varphi, \bar{\varphi}) = \frac{\partial^2}{\partial \varphi \partial \bar{\varphi}} K(\varphi, \bar{\varphi}), \quad (4.37)$$

and then (4.35) becomes

$$\left(\frac{\partial}{\partial t} + \eta + \Delta \right) Z(\varphi, \bar{\varphi}) = -\eta + W''(\varphi) \bar{W}''(\bar{\varphi}). \quad (4.38)$$

In contrast to scalar theories, Δ in (4.35) and (4.36) is just a straightforward derivative operator.

5 Discussion

The notion of the Wilsonian effective action plays a crucial role in the conceptual understanding of quantum field theories. Expressed in terms of local fields describing the relevant degrees of freedom it may be used to calculate physical amplitudes at energy scales ϵ for $\epsilon < \Lambda$ for Λ an energy cut off which is implicit in the effective action. The Wilson effective action is required to be quasi-local in the sense that expanding in terms of monomials in the fields there should be no singularities in the coefficient functions for momenta less than Λ . For application to a description of low energies ϵ the action may then be identified in terms of an expansion in terms of a basis of local operators, formed the fields and derivatives, consistent with the assumed symmetries of the theory, so long as there are no anomalies, with the expansion essentially a series in powers of ϵ/Λ .

Nevertheless a precise construction of the Wilsonian effective action is in general more elusive. For the restricted world of scalar field theories the Wilson/Polchinski exact RG equations offer a prescription for the effective action for an arbitrary continuously variable RG scale Λ in a form which lends itself to analytic treatment. It is also possible to formulate various approximation schemes although in general these lack systematic control. Many of these features can be extended to theories with fermion fields but gauge theories present major problems. A cut off restricted to the quadratic part of the initial action does not respect gauge invariance. Although various attempts have been made to extend exact RG equations to gauge theories they lack the essential simplicity of the scalar field equations and are also unable to demonstrate the presence of IR fixed points which are known to exist in very many quantum field theories in three and four dimensions. Moreover such RG equations do not manifestly describe the flow of a Wilsonian effective action without any long distance singularities.

At any IR fixed point with scale invariance there is expected to be also conformal symmetry. For conformal field theories the additional symmetry constrains the correlation function such that two and three point functions of conformal primary fields are fully determined up to an overall constant. It is natural to consider how conformal symmetry is realised in an exact RG framework, if only for scalar field theories. An extension of the Wilson exact RG equation to include conformal transformations was considered long ago [36]. In terms of the fixed point action S_* it is possible to define functional generators which satisfy

the algebra of the conformal group [37]. For any conformal field theory there is a natural scalar product defined by the two point function as a consequence of the state/operator correspondence. A precise definition of a scalar product such that the generator of scale transformations $\Delta_{\mathcal{S}^*, \text{loc}}$ is a hermitian operator would enable significant additional results for critical exponents to be obtained.

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A Redundant Operators and RG Flow

The results obtained in the text derive from the particular Polchinski RG equation. We show here how some results are also valid for more general RG equations in which Ψ_t in (2.11) is allowed to be essentially arbitrary.

For any RG equation there are redundant operators which correspond to redefinitions of the basic fields. We extend here the previous discussion to show that under RG flow these form a closed subspace. The flow equations are therefore applied to actions $\{S[\varphi]\}$ depending on a field φ and containing a cut off, which are invariant under the appropriate symmetry group and which are at most smooth deformations of local actions formed by integrals over an invariant action density \mathcal{L} constructed from a sum of monomials in the fields and derivatives at the same point. Such actions form a manifold \mathcal{M} which is in general infinite dimensional. Coordinates on \mathcal{M} may be identified with the couplings $\{g\}$. The basic flow equation, defining trajectory $S_t[\varphi] \in \mathcal{M}$, has the form

$$\frac{\partial}{\partial t} S_t = -D\varphi \cdot \frac{\delta}{\delta\varphi} S_t + \Psi_t \cdot \frac{\delta}{\delta\varphi} S_t - \frac{\delta}{\delta\varphi} \cdot \Psi_t, \quad (\text{A.1})$$

where we assume $\varphi \in V_\varphi$, the vector space formed by local fields, with a dual V_φ^* and \cdot then denotes the associated product on $V_\varphi \times V_\varphi^* \rightarrow \mathcal{V} = T\mathcal{M}_S$, the tangent space to \mathcal{M} at S . With $D : V_\varphi \rightarrow V_\varphi$ we require

$$\bar{\varphi} \cdot \varphi = \varphi \cdot \bar{\varphi}, \quad \bar{\varphi} \cdot D\varphi = \varphi \cdot \overleftarrow{D} \bar{\varphi} = \varphi \cdot \bar{D} \bar{\varphi}, \quad \varphi \in V_\varphi, \bar{\varphi} \in V_\varphi^*. \quad (\text{A.2})$$

Also in (A.1) $\Psi \in V_\varphi$ is here arbitrary except that we assume it has the functional form

$$\Psi\left(S, \frac{\delta}{\delta\varphi}\right), \quad (\text{A.3})$$

so that $\Psi_t = \Psi(S_t, \frac{\delta}{\delta\varphi})$. In (A.1) $S_t[\varphi]$ is arbitrary up to the addition of terms that do not depend on φ . This freedom may be used to absorb any φ -independent terms that may arise in any rearrangements of (A.1) giving alternative forms.

At a fixed point of the flow equation (A.1)

$$D\varphi \cdot \frac{\delta}{\delta\varphi} S_* - \Psi_* \cdot \frac{\delta}{\delta\varphi} S_* + \frac{\delta}{\delta\varphi} \cdot \Psi_* = 0, \quad (\text{A.4})$$

with $\Psi_* = \Psi(S_*, \frac{\delta}{\delta\varphi})$. In the neighbourhood of the fixed point we may write

$$S = S_* + \epsilon \mathcal{O}, \quad (\text{A.5})$$

for $\mathcal{O} \in \mathcal{V}_* = T\mathcal{M}_{S_*}$ the space of operators for the critical point theory. The variation (A.5) induces a change in Ψ of the form

$$\Psi\left(S, \frac{\delta}{\delta\varphi}\right) = \Psi_* + \epsilon \mathcal{D} \mathcal{O}, \quad (\text{A.6})$$

defining the linear operator $\mathcal{D} : \mathcal{V}_* \rightarrow V_\varphi$. Hence the operator $\Delta_{S_*} : \mathcal{V}_* \rightarrow \mathcal{V}_*$ given by

$$\Delta_{S_*} = D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} - \frac{\delta}{\delta\varphi} S_* \cdot \mathcal{D} + \frac{\delta}{\delta\varphi} \cdot \mathcal{D}. \quad (\text{A.7})$$

determines the critical exponents through the eigenvalue equation

$$\Delta_{S_*} \mathcal{O} = \lambda \mathcal{O}, \quad (\text{A.8})$$

as in (2.82) but for the more general Δ_{S_*} given by (A.7). Note that from (A.4)

$$\Delta_{S_*} \frac{\delta}{\delta\varphi} S_* = -\bar{D} \frac{\delta}{\delta\varphi} S_*. \quad (\text{A.9})$$

Redundant operators $\{\mathcal{O}_\psi\}$ here have the form

$$\mathcal{O}_\psi = \psi \cdot \frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi} \cdot \psi, \quad (\text{A.10})$$

for some $\psi \in V$ and defining a subspace $\mathcal{V}_R \subset \mathcal{V}_*$. In (A.10) the additional term present in (2.95) is omitted due to (2.20). The aim here is to show, as is necessary for such operators to form a closed space under RG flow, that $\Delta_{S_*} : \mathcal{V}_R \rightarrow \mathcal{V}_R$ or

$$\Delta_{S_*} \mathcal{O}_\psi = \mathcal{O}_{\tilde{\psi}}, \quad \tilde{\psi} = \tilde{\Delta}_{S_*} \psi, \quad (\text{A.11})$$

for some linear operator $\tilde{\Delta}_{S_*}$. To demonstrate (A.11) we first note that

$$\Delta_{S_*} \mathcal{O}_\psi = \left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \mathcal{O}_\psi - \mathcal{O}_{\mathcal{D}\mathcal{O}_\psi}. \quad (\text{A.12})$$

It is then sufficient to show, using (A.4),

$$\begin{aligned} & \left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \psi \cdot \frac{\delta}{\delta\varphi} S_* \\ &= \left(\left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \psi + \psi \cdot \frac{\delta}{\delta\varphi} \Psi_* - D\psi \right) \cdot \frac{\delta}{\delta\varphi} S_* - \psi \cdot \frac{\delta}{\delta\varphi} \left(\frac{\delta}{\delta\varphi} \cdot \Psi_* \right), \end{aligned} \quad (\text{A.13})$$

and also

$$\left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \frac{\delta}{\delta\varphi} \cdot \psi = \frac{\delta}{\delta\varphi} \cdot \left(\left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \psi - D\psi \right) + \left(\frac{\delta}{\delta\varphi} \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \cdot \psi, \quad (\text{A.14})$$

Since

$$\psi \cdot \frac{\delta}{\delta\varphi} \left(\frac{\delta}{\delta\varphi} \cdot \Psi_* \right) + \left(\frac{\delta}{\delta\varphi} \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \cdot \psi = \frac{\delta}{\delta\varphi} \cdot \left(\left(\psi \cdot \frac{\delta}{\delta\varphi} \right) \Psi_* \right), \quad (\text{A.15})$$

we then have in (A.11)

$$\tilde{\psi} = \left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi} \right) \psi + \psi \cdot \frac{\delta}{\delta\varphi} \Psi_* - \mathcal{D} \mathcal{O}_\psi - D\psi. \quad (\text{A.16})$$

determining $\tilde{\Delta}_{S_*}$ in (A.11).

For the particular case of Wilson/Polchinski equations it is sufficient to require

$$\Psi = \frac{1}{2} G \cdot \frac{\delta}{\delta\varphi} S, \quad \mathcal{D} = \frac{1}{2} G \cdot \frac{\delta}{\delta\varphi}, \quad (\text{A.17})$$

which leads to equations of the form (2.18). With this choice (A.16) then ensures that in (A.11)

$$\tilde{\Delta}_{S_*} = \Delta_{S_*} - D. \quad (\text{A.18})$$

For a zero mode corresponding to a redundant marginal operator we must have

$$\Delta_{S_*} \mathcal{Z} = 0, \quad \mathcal{Z} = \mathcal{O}_{\psi_{\mathcal{Z}}}, \quad \tilde{\Delta}_{S_*} \psi_{\mathcal{Z}} = 0. \quad (\text{A.19})$$

For the equations obtained by requiring (A.17) and (A.18), it is easy to construct such a $\psi_{\mathcal{Z}}$. Assuming

$$\psi_{\mathcal{Z}} = \varphi + \mathcal{H} \cdot \frac{\delta}{\delta \varphi} S_*, \quad (\text{A.20})$$

then using (A.9)

$$\tilde{\Delta}_{S_*} \psi_{\mathcal{Z}} = -(G + D \mathcal{H}) \cdot \frac{\delta}{\delta \varphi} S_* - \mathcal{H} \cdot \bar{D} \frac{\delta}{\delta \varphi} S_*, \quad (\text{A.21})$$

and therefore (A.19) is satisfied if

$$D \mathcal{H} + \mathcal{H} \overleftarrow{D} = -G. \quad (\text{A.22})$$

In solving (A.22) for \mathcal{H} it is necessary to impose the boundary conditions so that $\psi_{\mathcal{Z}}$ depends essentially locally on φ . In the context of the earlier discussion in 2.4 D takes the form $D\varphi = D^{(\delta)}\varphi + G \cdot \mathcal{G}^{-1} \cdot \varphi$, for δ as in (2.34), and then (A.22) becomes explicitly $(p \cdot \partial + 2 - 4p^2 K'(p^2)/K(p^2) - \eta) \mathcal{H}(p) = 2K'(p^2)$, which has a solution identical with that in (2.120).

A crucial issue in the discussion of exact RG equations such as (A.1) is the extent to which Ψ is restricted while maintaining an equation for the RG flow equation which allows non trivial IR fixed points to be realised as $t \rightarrow \infty$, even assuming (A.3). The trivial choice $\Psi = 0$ cannot lead to any non trivial fixed points, unlike the non linear equations obtained by taking (A.17).

In order to clarify such issues we consider infinitesimal variations in the functional form for Ψ in (A.3), $\delta\Psi(S, \frac{\delta}{\delta\varphi})$, leading to

$$S_* \rightarrow S_* + \delta S_* \quad \Rightarrow \quad \Psi_* \rightarrow \Psi_* + \mathcal{D}\delta S_* + \delta\Psi_*, \quad (\text{A.23})$$

with $\delta\Psi_* = \delta\Psi(S_*, \frac{\delta}{\delta\varphi})$. The variation of the fixed point equation (A.4) then gives

$$\Delta_{S_*} \delta S_* - \delta\Psi_* \cdot \frac{\delta}{\delta\varphi} S_* + \frac{\delta}{\delta\varphi} \cdot \delta\Psi_* = 0, \quad \text{or} \quad \Delta_{S_*} \delta S_* = \mathcal{O}_{\delta\Psi_*}. \quad (\text{A.24})$$

Special cases of (A.24) are given by (2.105) and (2.106). As a consequence of (A.11), (A.24) requires

$$\delta S_* = \mathcal{O}_{\psi}, \quad \tilde{\Delta}_{S_*} \psi = \delta\Psi_*. \quad (\text{A.25})$$

We here verify how perturbations such as (A.23), with δS_* a redundant operator given by (A.25), lead to equivalent results, as far as the eigenvalues determined by (A.8) are concerned, for apparently arbitrary choices for $\delta\Psi$. To show invariance of the spectrum

corresponding to non redundant operators \mathcal{O} in (2.103) it is sufficient to demonstrate that the operator \mathcal{O} can be modified to $\mathcal{O} + \delta\mathcal{O}$ such that

$$\Delta_{S_*} \delta\mathcal{O} + \delta\Delta_{S_*} \mathcal{O} = \lambda \delta\mathcal{O} + \mathcal{O}_\chi, \quad (\text{A.26})$$

for some suitable χ and where from (A.23)

$$\delta\Delta_{S_*} = -(\mathcal{D}\delta S_* + \delta\Psi_*) \cdot \frac{\delta}{\delta\varphi} - \left(\frac{\delta}{\delta\varphi}\delta S_*\right) \cdot \mathcal{D} - \frac{\delta}{\delta\varphi} S_* \cdot \delta\mathcal{D} + \frac{\delta}{\delta\varphi} \cdot \delta\mathcal{D}. \quad (\text{A.27})$$

The precise form for $\delta\mathcal{D}$, which is determined by extending (A.5) and (A.6), is subsequently irrelevant. The result (A.26) ensures that, to linear order, the perturbed eigenvalue equation is of the form (2.103) for the same eigenvalue λ as in the unperturbed case.

To achieve compatibility with (A.26) it is sufficient to take

$$\delta\mathcal{O} = \psi \cdot \frac{\delta}{\delta\varphi} \mathcal{O}, \quad (\text{A.28})$$

where ψ is determined by (A.25). In general, using (A.7) for Δ_{S_*} ,

$$\Delta_{S_*} \delta\mathcal{O} = \left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi}\right) \delta\mathcal{O} - \mathcal{O}_{\mathcal{D}\delta\mathcal{O}}, \quad (\text{A.29})$$

and the basic eigenvalue equation (A.8) with (A.16) ensures that

$$\begin{aligned} \left(D\varphi \cdot \frac{\delta}{\delta\varphi} - \Psi_* \cdot \frac{\delta}{\delta\varphi}\right) \left(\psi \cdot \frac{\delta}{\delta\varphi} \mathcal{O}\right) &= \lambda \psi \cdot \frac{\delta}{\delta\varphi} \mathcal{O} + \psi \cdot \frac{\delta}{\delta\varphi} \left(\left(\frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi}\right) \cdot \mathcal{D}\mathcal{O}\right) \\ &\quad + (\tilde{\Delta}_{S_*} \psi + \mathcal{D}\mathcal{O}_\psi) \cdot \frac{\delta}{\delta\varphi} \mathcal{O}. \end{aligned} \quad (\text{A.30})$$

Furthermore from (A.27)

$$\delta\Delta_{S_*} \mathcal{O} = -(\mathcal{D}\mathcal{O}_\psi + \delta\Psi_*) \cdot \frac{\delta}{\delta\varphi} \mathcal{O} - \left(\frac{\delta}{\delta\varphi} \mathcal{O}_\psi\right) \cdot \mathcal{D}\mathcal{O} - \mathcal{O}_{\delta\mathcal{D}\mathcal{O}}. \quad (\text{A.31})$$

Hence combining (A.29), (A.30) and (A.31), and using (A.25),

$$\begin{aligned} \Delta_{S_*} \delta\mathcal{O} + \delta\Delta_{S_*} \mathcal{O} - \lambda \delta\mathcal{O} &= -\mathcal{O}_{\delta\mathcal{D}\mathcal{O}} - \mathcal{O}_{\mathcal{D}\delta\mathcal{O}} \\ &\quad - \frac{\delta}{\delta\varphi} \mathcal{O}_\psi \cdot \mathcal{D}\mathcal{O} + \psi \cdot \frac{\delta}{\delta\varphi} \left(\left(\frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi}\right) \cdot \mathcal{D}\mathcal{O}\right). \end{aligned} \quad (\text{A.32})$$

This may be simplified by virtue of

$$\psi \cdot \frac{\delta}{\delta\varphi} \left(\left(\frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi}\right) \cdot \mathcal{D}\mathcal{O}\right) - \frac{\delta}{\delta\varphi} \left(\psi \cdot \frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi} \psi\right) \cdot \mathcal{D}\mathcal{O} = \xi \cdot \frac{\delta}{\delta\varphi} S_* - \frac{\delta}{\delta\varphi} \cdot \xi, \quad (\text{A.33})$$

where

$$\xi = \psi \cdot \frac{\delta}{\delta\varphi} \mathcal{D}\mathcal{O} - \mathcal{D}\mathcal{O} \cdot \frac{\delta}{\delta\varphi} \psi. \quad (\text{A.34})$$

This then ensures that (A.26) is satisfied if we take

$$\chi = \xi - \delta\mathcal{D}\mathcal{O} - \mathcal{D}\delta\mathcal{O}. \quad (\text{A.35})$$

If arbitrary variations $\delta\Psi$ were allowed then it would be possible to continuously transform Ψ_* to zero, contradicting the expected triviality of the RG equations for $\Psi = 0$. However there are potential obstructions to unconstrained variations $\delta\Psi_*$ at non trivial fixed points due to the presence of zero modes satisfying (A.19). To show this we assume there is a scalar product with respect to which Δ_{S_*} is hermitian so that

$$\langle \mathcal{O}', \Delta_{S_*} \mathcal{O} \rangle = \langle \Delta_{S_*} \mathcal{O}', \mathcal{O} \rangle. \quad (\text{A.36})$$

From this it follows that the critical exponents defined (A.8) must be real. As a direct consequence of (A.36), (A.24) also requires that $\delta\Psi$ must satisfy

$$\langle \mathcal{Z}, \mathcal{O}_{\delta\Psi_*} \rangle = 0. \quad (\text{A.37})$$

This requires that $\delta\Psi_*$ is restricted to a surface of codimension one in the space of all possible variations. Such a condition should ensure that the anomalous dimension η , which was initially introduced in terms of a contribution to Ψ , cannot be varied at the critical point when $\Psi \rightarrow \Psi_*$. This quantisation of η is then directly associated with the presence of the zero mode \mathcal{Z} .

B Perturbative Considerations

Any exact RG equation should of course generate the usual perturbative results in a weak coupling expansion. This requires that at a fixed point, if this is accessible perturbatively as in the ε -expansion, the critical exponents found by using an exact RG equation should match those found in an ε -expansion. Of course with approximations this may no longer hold but may perhaps be used as a form of boundary condition to constrain any free parameters which appear for example in a derivative expansion. That lowest order perturbative results can be matched with those obtained in a derivative expansion is demonstrated in this section although extending this beyond lowest order is more problematic.

For scalar theories we consider therefore the simple Lagrangian

$$\mathcal{L}_V = \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi + V(\phi). \quad (\text{B.1})$$

Formally for dimensions $d_n = 2(n+1)/n$, $n = 1, 2, \dots$, this has a renormalisable perturbative expansion if $V(\phi)$ is a polynomial of degree $2(n+1)$ which may then be expressed in terms of a finite linear sum $\sum_I g^I \mathcal{O}_I(\phi)$ over all linearly independent monomials $\mathcal{O}_I(\phi)$, with coefficients the couplings g^I parameterising the renormalisable theory. Renormalisability ensures that the β -functions determining the perturbative RG flow are also expressible in terms of a similar expansion $\beta_V(\phi) = \sum_I \beta^I(g) \mathcal{O}_I(\phi)$. Using dimensional regularisation and including the canonical dimension then the RG flow becomes

$$\frac{d}{dt} V(\phi) = -B_V(\phi), \quad (\text{B.2})$$

where

$$B_V(\phi) = (\Gamma_\phi \phi) \cdot \partial V(\phi) - dV(\phi) + \tilde{\beta}_V(\phi), \quad \Gamma_{\phi,ij} = \frac{1}{2}(d-2)\delta_{ij} + \gamma_{\phi,ij} \quad (\text{B.3})$$

In (B.3) the contribution to the β -functions involving the anomalous dimension matrix, $\gamma_{\phi,ij}$, for ϕ , has been isolated, the usual β -function is then $\beta_V(\phi) = \tilde{\beta}_V(\phi) + (\gamma_\phi \phi) \cdot \partial V(\phi)$. Using dimensional regularisation by letting $d = d_n - \varepsilon$, and assuming minimal subtraction of poles in ε , $\tilde{\beta}_V(\phi)$ does not depend on d explicitly and in each order of a loop expansion $\tilde{\beta}_V(\phi)$ is a scalar polynomial formed from contractions of $V_{i_1 i_2 \dots i_k}(\phi) = \partial_{i_1} \dots \partial_{i_k} V(\phi)$ with $k \geq 2$. Fixed points arise when $V = V_*$ if

$$B_{V_*}(\phi) = 0. \quad (\text{B.4})$$

For discussion of the critical exponents which are associated with mixing of scalar non derivative and derivative operators the fundamental Lagrangian is extended to

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_{F,G}, \quad \mathcal{L}_{F,G} = F(\phi) + \frac{1}{2} G_{ij}(\phi) \partial^\mu \phi_i \partial_\mu \phi_j, \quad (\text{B.5})$$

with \mathcal{L}_V as in (B.1) and, as discussed above, $V(\phi)$ a polynomial of degree $2(n+1)$. The theory defined by (B.5) is renormalisable, in the sense that all counterterms linear in F, G_{ij} , may be absorbed into a bare Lagrangian \mathcal{L}_0 which has the same functional form as (B.5) if, in dimensions d_n , $F(\phi)$ is a polynomial in ϕ of degree $4n-1$ and $G_{ij}(\phi)$ also a polynomial of degree $2n-1$. For higher degree four derivative terms are also necessary in (B.5).

In a similar fashion to the definition of the usual β -functions we may define linear operators acting on F, G_{ij} ,

$$\begin{aligned} \Delta_V \begin{pmatrix} F(\phi) \\ G_{ij}(\phi) \end{pmatrix} \\ = \begin{pmatrix} (\Gamma_\phi \phi) \cdot \partial F(\phi) - dF(\phi) + \gamma_{FF} F(\phi) + \gamma_{FG,ij} G_{ij}(\phi) \\ (\Gamma_\phi \phi) \cdot \partial G_{ij}(\phi) + \gamma_{\phi,ik} G_{kj}(\phi) + \gamma_{\phi,jk} G_{ik}(\phi) + \gamma_{GF,ij} F(\phi) + \gamma_{GG,ijkl} G_{kl}(\phi) \end{pmatrix}, \end{aligned} \quad (\text{B.6})$$

with Γ_ϕ defined as in (B.3) and $\gamma_{FF}, \gamma_{FG,ij}, \gamma_{GF,ij}, \gamma_{GG,ijkl}$ are differential operators depending on the couplings g^I or V . If $F(\phi)$ is restricted to be a polynomial of degree $2(n+1)$, then

$$\tilde{\beta}_{V+F}(\phi) = \beta_V(\phi) + \gamma_{FF} F(\phi) + \mathcal{O}(F^2). \quad (\text{B.7})$$

At a fixed point the exponents are defined by the coupled linear equations

$$\Delta_{V_*} \begin{pmatrix} F(\phi) \\ G_{ij}(\phi) \end{pmatrix} = -\lambda \begin{pmatrix} F(\phi) \\ G_{ij}(\phi) \end{pmatrix}. \quad (\text{B.8})$$

The action defined by the lagrangian in (B.5) is invariant under

$$\delta \mathcal{L}_V = \delta_{F,G} \mathcal{L}_{F,G} \quad \text{for} \quad \delta \phi_i = v_i(\phi), \quad \delta \partial_\mu \phi_i = v_{i,j}(\phi) \partial_\mu \phi_j, \quad (\text{B.9})$$

if

$$\delta_{F,G} F(\phi) = v(\phi) \cdot \partial V(\phi), \quad \delta_{F,G} G_{ij}(\phi) = \partial_i v_j(\phi) + \partial_j v_i(\phi). \quad (\text{B.10})$$

Assuming $F(\phi), G_{ij}(\phi)$ are restricted to ensure that no mixing with four derivative operators arises then it is necessary to require $v_i(\phi)$ is a polynomial in ϕ of degree $2n$. In [31] it was shown how this leads to identities for Δ_V such that

$$\Delta_V \begin{pmatrix} v(\phi) \cdot \partial V(\phi) \\ \partial_i v_j(\phi) + \partial_j v_i(\phi) \end{pmatrix} = \begin{pmatrix} U(\phi) \cdot \partial V(\phi) + v(\phi) \cdot \partial B_V(\phi) \\ \partial_i U_j(\phi) + \partial_j U_i(\phi) \end{pmatrix}, \quad (\text{B.11})$$

for

$$U_i(\phi) = (\Gamma_\phi \phi) \cdot \partial v_i(\phi) + \gamma_{FF} v_i(\phi) - \Gamma_{\phi,ij} v_j(\phi). \quad (\text{B.12})$$

The result (B.11) ensures that at a critical point, where (B.4) holds, there are solutions of (B.8) if

$$U_i(\phi) = -\lambda v_i(\phi). \quad (\text{B.13})$$

In particular since γ_{FF} involves at least second order derivatives with respect to ϕ there are exact zero modes, with $\lambda = 0$, obtained by taking $v_i(\phi) \rightarrow \phi_i$,

$$F_0(\phi) = \phi \cdot \partial V(\phi), \quad G_{0,ij} = 2\delta_{ij}. \quad (\text{B.14})$$

In perturbative calculations (B.11) is equivalent to

$$\gamma_{FF}(v(\phi) \cdot \partial V(\phi)) + \gamma_{FG,ij}(\partial_i v_j(\phi) + \partial_j v_i(\phi)) = (\gamma_{FF} v(\phi)) \cdot \partial V(\phi) + v(\phi) \cdot \partial \tilde{\beta}_V(\phi), \quad (\text{B.15})$$

and

$$\begin{aligned} & \gamma_{GF,ij}(v(\phi) \cdot \partial V(\phi)) + \gamma_{GG,ijkl}(\partial_k v_l(\phi) + \partial_l v_k(\phi)) \\ &= \partial_i(\gamma_{FF} v_j(\phi) - 2\gamma_{\phi,jk} v_k(\phi)) + \partial_j(\gamma_{FF} v_i(\phi) - 2\gamma_{\phi,ik} v_k(\phi)). \end{aligned} \quad (\text{B.16})$$

For $d = d_n$ there are logarithmic divergencies at pn loops for $p = 1, 2, \dots$ which give rise to contributions to the β -functions. At lowest order for the β -function associated with V

$$\tilde{\beta}_V^{(n)}(\phi) = a_n V_{i_1 \dots i_{n+1}}(\phi) V_{i_1 \dots i_{n+1}}(\phi), \quad (\text{B.17})$$

from which it is easy to see that

$$\gamma_{FF}^{(n)} F(\phi) = 2a_n V_{i_1 \dots i_{n+1}}(\phi) F_{i_1 \dots i_{n+1}}(\phi). \quad (\text{B.18})$$

We also have [18, 31]

$$\begin{aligned} \gamma_{FG,ij}^{(n)} G_{ij}(\phi) &= -a_n \sum_{\substack{r,s,t \geq 1 \\ r+s+t=n+2}} \frac{(n+1)!}{r!s!t!} \hat{K}_{rst} V_{i_1 \dots i_r k_1 \dots k_t}(\phi) V_{j_1 \dots j_s k_1 \dots k_t}(\phi) G_{i_1 j_1, i_2 \dots i_r j_2 \dots j_s}(\phi) \\ &+ a_n \sum_{\substack{r \geq 2, s, t \geq 1 \\ r+s+t=n+2}} \frac{(n+1)!}{r!s!t!} \hat{K}_{rst} V_{i_1 \dots i_r k_1 \dots k_t}(\phi) V_{j_1 \dots j_s k_1 \dots k_t}(\phi) G_{i_1 i_2, i_3 \dots i_r j_1 \dots j_s}(\phi), \end{aligned} \quad (\text{B.19})$$

and

$$\begin{aligned} \gamma_{GG,ijkl}^{(n)} G_{kl}(\phi) &= 2a_n (V_{i_1 \dots i_{n+1}}(\phi) G_{ij, i_1 \dots i_{n+1}}(\phi) + 2V_{i_1 \dots i_{n+1}}(\phi) G_{j(i_1, i_2 \dots i_{n+1}}(\phi) \\ &\quad - V_{i_1 \dots i_{n+1}}(\phi) G_{i_1 i_2, j)(i_3 \dots i_{n+1}}(\phi)), \end{aligned} \quad (\text{B.20a})$$

$$\gamma_{FG,kl}^{(n)} G_{kl}(\phi) = 0. \quad (\text{B.20b})$$

The precise expressions for a_n and \hat{K}_{rst} are given in [31] but are unimportant here, it is only necessary to note that \hat{K}_{rst} is symmetric and satisfies

$$\hat{K}_{1st} = 1, \quad s + t = n + 1. \quad (\text{B.21})$$

With these results it is straightforward to verify that (B.15) and (B.16) are satisfied since $\gamma_{\phi,ij}^{(n)} = 0$ at this order.

At the next order

$$\begin{aligned}\tilde{\beta}_V^{(2n)}(\phi) &= -\frac{b_n}{3n} \sum_{\substack{r,s,t \geq 1 \\ r+s+t=2n+2}} \frac{(2n+2)!}{r!s!t!} \hat{K}_{rst} V_{i_1 \dots i_r j_1 \dots j_s}(\phi) V_{i_1 \dots i_r k_1 \dots k_t}(\phi) V_{j_1 \dots j_s k_1 \dots k_t}(\phi), \\ \gamma_{\phi,ij}^{(2n)} &= 2b_n g_{i i_1 \dots i_{2n+1}} g_{j i_1 \dots i_{2n+1}}\end{aligned}\tag{B.22}$$

where $g_{i_1 \dots i_{2n+2}} = V_{i_1 \dots i_{2n+2}}(\phi)$ is the dimensionless coupling and $b_n = ((n+1)!)^2 a_n^2 / (2n+2)!$. \hat{K}_{rst} includes the effect of subdivergencies when any r, s, t are equal to $n+1$.

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